



Galois Deformation Ring and Barsotti-Tate Representations in the Relative Case

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Galois Deformation Ring and Barsotti-Tate Representations in the Relative Case

A dissertation presented

by

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to

The Department of Mathematics

in partial fulfillment of the requirements

for the degree of

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in the subject of

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Abstract

In this thesis, we study finite locally free group schemes, Galois deformation rings, and Barsotti-Tate representations in the relative case. We show three independent but related results, assuming $p > 2$.

First, we give a simpler alternative proof of Breuil's result on classifying finite flat group schemes over the ring of integers of a p -adic field by certain Breuil modules [5]. Second, we prove that the locus of potentially semi-stable representations of the absolute Galois group of a p -adic field K with a specified Hodge-Tate type and Galois type cuts out a closed subspace of the generic fiber of a given Galois deformation ring, without assuming that K/\mathbb{Q}_p is finite. This is an extension of the corresponding result of Kisin when K/\mathbb{Q}_p is finite [19]. Third, we study the locus of Barsotti-Tate representations in the relative case, via analyzing certain extendability of p -divisible groups. We prove that when the ramification index is less than $p - 1$, the locus of relative Barsotti-Tate representations cuts out a closed subspace of the generic fiber of a Galois deformation ring, if the base scheme is 2-dimensional satisfying some conditions. When the ramification index is greater than $p - 1$, we show that such a result does not hold in general.

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1 Introduction

1.1 Motivation

A remarkable relationship between representations of the absolute Galois group of a number field and algebraic geometry has been predicted by Fontaine-Mazur conjecture [14]. It predicts that any global irreducible p -adic Galois representation which is potentially semi-stable at primes dividing p and unramified outside finitely many places is a subquotient of the p -adic cohomology of algebraic variety, up to Tate twist. The conjecture holds for 1-dimensional case essentially by class field theory and the theory of CM abelian varieties, and most of the 2-dimensional cases have been proved by the works of Kisin [20] and Emerton [10] based on modularity lifting theorems.

Mazur showed that the moduli spaces of Galois representations are represented by universal deformation rings [30]. To understand which Galois representations come from algebraic geometry, it is essential to study the sub-moduli spaces which parametrize those with certain p -adic Hodge theoretic conditions. For example, Kisin proved that the locus of representations which are potentially semi-stable with a given Hodge-Tate type and Galois type cuts out a closed subscheme of the generic fiber of the universal deformation ring [19], and studied this subspace to prove his result in [20] for the 2-dimensional case.

On the other hand, little has been studied in this direction for the Galois groups of higher dimensional rings of arithmetic interest over number fields. For a smooth proper geometrically connected scheme of finite type over a p -adic field, the moduli space of continuous representations of the étale fundamental group is represented by a universal deformation ring [30]. A main part of this thesis is the study of the sub-moduli space of *Barsotti-Tate representations in the relative case*.

Let K be a p -adic field, and let \mathcal{X} be a (formally) smooth geometrically connected

scheme over the ring of integers \mathcal{O}_K . We assume $p > 2$ throughout. Denote by \mathcal{G} the étale fundamental group of $\mathcal{X} \times_{\mathcal{O}_K} K$. We say that a \mathbb{Q}_p -representation V is *Barsotti-Tate* if there exists a p -divisible group $G_{\mathcal{X}}$ over \mathcal{X} such that $T_p(G_{\mathcal{X}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong V$ as representations of \mathcal{G} , where $T_p(G_{\mathcal{X}})$ denotes the Tate module associated to $G_{\mathcal{X}}$. For a \mathbb{Z}_p -representation L of \mathcal{G} , we say it is *Barsotti-Tate* if $L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is Barsotti-Tate. Barsotti-Tate representations are of particular interest since they are crystalline having Hodge-Tate weights in $\{0, 1\}$ in the sense of [6], and moreover when $\mathcal{X} = \text{Spec} \mathcal{O}_K$ any crystalline representation of Hodge-Tate weights in $\{0, 1\}$ is Barsotti-Tate [18]. We study the locus of Barsotti-Tate representations via studying torsion representations which are quotients of Barsotti-Tate representations and applying the Schlessinger's criterion [33]. More precisely, we study the following question which is slightly stronger than asking whether being Barsotti-Tate is a Zariski-closed condition.

Question 1.1. Let L be a \mathbb{Z}_p -representation of \mathcal{G} such that L/p^n is a quotient of Barsotti-Tate \mathbb{Z}_p -representations for all positive integers n . Then is L a Barsotti-Tate representation?

As stated in the next section, we prove that when the ramification index e for K is small, Question 1.1 has an affirmative answer for \mathcal{X} of dimension 2 satisfying certain assumptions. As a corollary, we prove in this case that the locus of Barsotti-Tate representations cuts out a closed subscheme. When e is large, we show that there exist counter-examples even in the most simple relative cases. Since we naturally obtain from a representation of \mathcal{G} a family of representations of the absolute Galois groups of p -adic fields via base changes, we expect to see some applications to Fontaine-Mazur conjecture.

We also present in this thesis two more results which are independent but related to above. First, we give a simpler alternative proof for Breuil's classification of finite

flat group schemes over \mathcal{O}_K by certain Breuil modules [5]. Second, we prove that being potentially semi-stable with a given Hodge-Tate type and Galois type is a Zariski-closed condition, when the residue field of K is only assumed to be perfect. This generalizes the corresponding result by Kisin when the residue field is finite [19].

1.2 Statement of the Results

Let k be a perfect field of characteristic $p > 2$, and let $W(k)$ denote its ring of Witt vectors. Let K be a finite totally ramified extension of $\text{Frac}(W(k))$, and denote by \mathcal{O}_K its ring of integers. Let $E(u)$ be the Eisenstein polynomial for the extension $K/\text{Frac}(W(k))$. We fix a uniformizer ϖ of \mathcal{O}_K . In this section, we state briefly the three independent results of the thesis.

1.2.1 Breuil's Classification of Finite Flat Group Schemes

We give a simpler alternative proof of Breuil's result on classifying finite flat group schemes over \mathcal{O}_K [5, Theorem 4.2.1.6]. Let $\mathfrak{S} = W(k)[[u]]$ be the formal power series ring, and let S be the p -adic completion of the divided power envelope of \mathfrak{S} with respect to the ideal $(E(u))$. Let $\text{Fil}^1 S \subset S$ be the p -adic completion of the ideal generated by the divided powers $\frac{E(u)^i}{i!}$ with $i \geq 1$. Denote by φ the Frobenius map on \mathfrak{S} given by the natural Frobenius on $W(k)$ and $\varphi(u) = u^p$, which has the unique continuous extension to S . Breuil considered the category (Mod/S) of p -power torsion S -modules equipped with a one-step filtration and φ -semilinear map satisfying certain properties (cf. Section 2.2 for the precise definition), and showed the following.

Theorem 1.2. (Breuil [5, Theorem 4.2.1.6]) *The category of finite flat group schemes over \mathcal{O}_K is anti-equivalent to (Mod/S) .*

Breuil proved Theorem 1.2 by working over the syntomic topology introduced by Mazur in [29] and associating the objects in (Mod/S) free over S/p to group schemes killed by p . In this thesis, we give a simpler alternative proof of Theorem 1.2.

1.2.2 Potentially Semi-stable Deformation Ring

We generalize the result in [19] that certain p -adic Hodge theoretic conditions correspond to closed algebraic conditions, to the case when the residue field k of K is not necessarily finite. We fix an algebraic closure \bar{K} of K , and let $\mathcal{G}_K := \text{Gal}(\bar{K}/K)$ be the absolute Galois group of K . Let E/\mathbb{Q}_p be a finite extension with residue field \mathbb{F} , and denote by \mathcal{O}_E its ring of integers. Let A_0 be a complete noetherian local \mathcal{O}_E -algebra having residue field \mathbb{F} . Let V_0 be a finite \mathcal{G}_K -representation over A_0 , and we fix a p -adic Hodge-Tate type \mathbf{v} and Galois type τ . Then, we show the following theorem.

Theorem 1.3. *There exists a quotient $A_0^{\mathbf{v},\tau}$ such that for any finite E -algebra B , an E -algebra morphism $\rho : A_0[\frac{1}{p}] \rightarrow B$ factors through $A_0^{\mathbf{v},\tau}[\frac{1}{p}]$ if and only if the induced representation $V_0 \otimes_{A_0,\rho} B$ is potentially semi-stable having Hodge-Tate type \mathbf{v} and Galois type τ .*

1.2.3 Barsotti-Tate Representations in the Relative Case

As above, let \mathcal{X} be a (formally) smooth geometrically connected scheme over \mathcal{O}_K . To investigate Question 1.1, we study the following very closely related question.

Question 1.4. Let $G = (G_n)_{n \geq 1}$ be a p -divisible group over $\mathcal{X} \times_{\mathcal{O}_K} K$. Suppose that for each n , G_n extends to a finite locally free group scheme over \mathcal{X} . Then does G extend to a p -divisible group over \mathcal{X} ?

When $\mathcal{X} = \operatorname{Spec} \mathcal{O}_K$, Question 1.4 can be seen to be equivalent to Question 1.1 since any finite locally free group scheme over \mathcal{O}_K embeds into a p -divisible group by [2, Theorem 3.1.1], and Raynaud showed that it has an affirmative answer [32]. We prove that the answer to Question 1.4 is negative when the ramification index e for K is large, even for the simplest relative cases.

Theorem 1.5. *When $e \geq p$ and $\mathcal{X} = \operatorname{Spec} \mathcal{O}_K[[t]]$, a p -divisible group G given as in Question 1.4 does not extend to \mathcal{X} in general.*

Instead, we prove that the following weaker extendability holds.

Theorem 1.6. *Let $\mathcal{X} = \operatorname{Spec} \mathcal{O}_K[[t]]$ and let G be as in Question 1.4. Then G extends to a p -divisible group over $\mathcal{X} \setminus \{\mathfrak{m}\}$, where \mathfrak{m} is the maximal ideal of $\mathcal{O}_K[[t]]$. Furthermore, such an extension is unique up to isomorphism.*

When the ramification index e is low, we prove that Question 1.4 has the positive answer for many cases. To explain the result, we need the following definition.

Definition 1.7. We say that a ring R_0 is *unramified-good* if there exists a perfect field k' over k and a finitely generated ideal $J \subset R_0$ containing p such that R_0 is a formally smooth formally finite type $W(k')$ -algebra with respect to the J -adic topology. We say that a ring R is *good* if $R \cong R_0 \otimes_{W(k)} \mathcal{O}_K$ for some unramified-good ring R_0 .

Theorem 1.8. *Let G be as in Question 1.4.*

1. *Let $R = \mathcal{O}_K[[t]]$ or $R = \mathcal{O}_K\langle t^{\pm 1} \rangle$ (the p -adic completion of $\mathcal{O}_K[t^{\pm 1}]$), and let $\mathcal{X} = \operatorname{Spec} R$. Suppose that $e \leq p-1$. Then G extends to a p -divisible group over \mathcal{X} uniquely up to isomorphism.*

2. Assume $e < p - 1$. Suppose that \mathcal{X} can be covered by finitely many affine open subschemes $\text{Spec} A_i$ such that A_i is good as in Definition 1.7. Then G extends to a p -divisible group over \mathcal{X} .

As an application, we give the answer to Question 1.1 when \mathcal{X} has dimension 2 satisfying certain conditions. Let \mathcal{G} be the étale fundamental group of $\mathcal{X} \times_{\mathcal{O}_K} K$.

Theorem 1.9. 1. Suppose $e < p - 1$, and \mathcal{X} has dimension 2 and satisfies the assumption as in Theorem 1.8(2). Let L be a \mathbb{Z}_p -representation of \mathcal{G} . Suppose that for each positive integer n , L/p^n is torsion Barsotti-Tate, in the sense that there exist \mathcal{G} -stable \mathbb{Z}_p -lattices $T'_n \subset T_n$ inside a Barsotti-Tate \mathbb{Q}_p -representation such that $L/p^n \cong T_n/T'_n$ as $\mathbb{Z}_p[\mathcal{G}]$ -modules. Then L is Barsotti-Tate.

2. Let $e \geq p$ and $\mathcal{X} = \text{Spec} \mathcal{O}_K[[t]]$. Then there exists a \mathbb{Z}_p -representation L of \mathcal{G} such that L/p^n is torsion Barsotti-Tate for each positive integer n but L is not Barsotti-Tate.

As a corollary, we have the following.

Corollary 1.10. Suppose $e < p - 1$, and \mathcal{X} has dimension 2 and satisfies the assumption as in Theorem 1.8(2). Then being Barsotti-Tate as a representation of \mathcal{G} is a Zariski-closed condition.

1.3 Structure of the Thesis

We present the three results in Section 1.2 in separate sections. Section 2 is on Breuil's classification of finite flat group schemes (cf. Section 1.2.1). Section 3 is on potentially semi-stable deformation rings (cf. Section 1.2.2). Lastly, Section 4 is on Barsotti-Tate representations in the relative case (cf. Section 1.2.3).

2 Breuil's Classification of Finite Flat Group Schemes

2.1 Introduction

We keep the notation as in Section 1. We give a simpler alternative proof to Breuil's classification of finite flat group schemes (Theorem 1.2), based on the following two observations.

Using Grothendieck-Messing theory, Kisin showed in [18, Appendix A] that there is an exact contravariant functor from the category of p -divisible groups over \mathcal{O}_K to the category $\mathrm{BT}_S(\varphi)$ of finite free S -modules with a filtration and Frobenius structure satisfying certain properties (cf. Section 2.2 for the precise definition), and showed it is an anti-equivalence. We remark that such anti-equivalence is also proved in [5] using Theorem 1.2, but the direct computation in [18] is significantly simpler.

On the other hand, any finite flat group scheme over \mathcal{O}_K is the kernel of an isogeny of p -divisible groups over \mathcal{O}_K (cf. [2, Theorem 3.1.1]). This allows us to associate a finite flat group scheme over \mathcal{O}_K with the cokernel of an isogeny of objects in $\mathrm{BT}_S(\varphi)$, which can be shown to lie in (Mod/S) . We further show that any object of (Mod/S) is the cokernel of an isogeny in $\mathrm{BT}_S(\varphi)$ and prove Theorem 1.2.

2.2 Properties of Breuil-Kisin Modules

Let e be the degree of the extension $K/\mathrm{Frac}W(k)$. We first introduce some categories of Breuil modules considered in [5]. For each positive integer j , let $\mathrm{Fil}^j S$ be the p -adic completion of the ideal of S generated by the divided powers $\frac{E(u)^i}{i!}$ with $i \geq j$. The Frobenius φ on S satisfies $\varphi(\mathrm{Fil}^1 S) \subset pS$. Denote $\varphi_1 = \frac{\varphi}{p}|_{\mathrm{Fil}^1 S}$. Let $c = \varphi_1(E(u))$ which is a unit in S . Let $'(\mathrm{Mod}/S)$ denote the category whose objects are triples $(M, \mathrm{Fil}^1 M, \varphi_1)$ consisting of:

- an S -module M ,
- S -submodule $\text{Fil}^1 M \subset M$ containing $\text{Fil}^1 S \cdot M$, and
- φ -semilinear map $\varphi_1 : \text{Fil}^1 M \rightarrow M$ such that for any $s \in \text{Fil}^1 S$ and $x \in M$, we have $\varphi_1(sx) = c^{-1}\varphi_1(s)\varphi_1(E(u)x)$.

Morphisms in $'(\text{Mod}/S)$ are morphisms of S -modules respecting the filtration Fil^1 's and φ_1 . We say a sequence of morphisms in $'(\text{Mod}/S)$ is exact if it is exact as a sequence of S -modules and induces an exact sequence on Fil^1 's. For $S_1 := S/pS$, we denote by (Mod/S_1) the full subcategory of $'(\text{Mod}/S)$ consisting of objects which are annihilated by p such that M is finite free over S_1 and $\varphi_1(\text{Fil}^1 M)$ generates M over S . Let (Mod/S) denote the smallest full subcategory of $'(\text{Mod}/S)$ containing the objects of (Mod/S_1) and stable under extensions. The following lemmas are proved in [5].

Lemma 2.1. ([5, Lemma 2.1.1.1]) *Let $M \in (\text{Mod}/S)$. Then M is a finite S -module killed by some power of p , and $\varphi_1(\text{Fil}^1 M)$ generates M over S .*

Lemma 2.2. ([5, Proposition 2.1.2.5]) *Let $M \in (\text{Mod}/S_1)$ be of rank d over S_1 . Then there exist a basis $\{e_1, \dots, e_d\}$ of M and integers $r_1, \dots, r_d \in \{0, \dots, e\}$ such that $(u^{r_1}e_1, \dots, u^{r_d}e_d) + \text{Fil}^p S_1 \cdot M$ generates $\text{Fil}^1 M$.*

Corollary 2.3. *Let $N \in (\text{Mod}/S)$. Then there exist elements $e_1, \dots, e_d \in N$ and $d_1 \in \{1, \dots, d\}$ such that*

$$N = \sum_{i=1}^d S e_i, \quad \text{and} \quad \text{Fil}^1 N = \sum_{i=1}^{d_1} S e_i + \sum_{i=d_1+1}^d \text{Fil}^1 S \cdot e_i.$$

Proof. Since N is obtained from successive extensions by objects in (Mod/S_1) , it follows from Lemma 2.2 that there exist $f_1, \dots, f_l \in \text{Fil}^1 N$ such that $\text{Fil}^1 N = \sum_{i=1}^l S f_i + \text{Fil}^1 S \cdot N$.

N is finite over S by Lemma 2.1, so including $\{f_1, \dots, f_l\}$ to a set of generators of N gives the result. \square

Let $\text{BT}_S(\varphi)$ denote the full subcategory of $'(\text{Mod}/S)$ consisting of objects such that M is finite free over S , $M/\text{Fil}^1 M$ is p -torsion free, and $\varphi_1(\text{Fil}^1 M)$ generates M over S .

Lemma 2.4. ([5, Lemma 2.1.1.9]) *Let $M \in \text{BT}_S(\varphi)$. Then there exist a basis $\{e_1, \dots, e_d\}$ of M and $d_1 \in \{1, \dots, d\}$ such that*

$$\text{Fil}^1 M = \left(\bigoplus_{i=1}^{d_1} S e_i \right) \oplus \left(\bigoplus_{i=d_1+1}^d \text{Fil}^1 S \cdot e_i \right).$$

In [18], Kisin considered certain categories of \mathfrak{S} -modules equipped with φ -semilinear endomorphisms. We can naturally associate such modules to the modules over S given above. Let $'(\text{Mod}/\mathfrak{S})$ denote the category of \mathfrak{S} -modules \mathfrak{M} equipped with a φ -semilinear map $\varphi : \mathfrak{M} \rightarrow \mathfrak{M}$ such that the cokernel of $1 \otimes \varphi : \varphi^*(\mathfrak{M}) \rightarrow \mathfrak{M}$ is killed by $E(u)$. The morphisms in $'(\text{Mod}/\mathfrak{S})$ are morphisms of \mathfrak{S} -modules compatible with φ . Let $(\text{Mod}/\mathfrak{S})$ be the smallest full subcategory of $'(\text{Mod}/\mathfrak{S})$ containing the objects which are finite free over \mathfrak{S}/p and stable under extensions. Denote by $\text{BT}_{\mathfrak{S}}(\varphi)$ the full subcategory of $'(\text{Mod}/\mathfrak{S})$ consisting of objects finite free over \mathfrak{S} .

We construct a functor $F : \text{BT}_{\mathfrak{S}}(\varphi) \rightarrow '(\text{Mod}/S)$ in the following way. Given $\mathfrak{M} \in \text{BT}_{\mathfrak{S}}(\varphi)$, we set $M = S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ and

$$\text{Fil}^1 M = \{x \in M \mid (1 \otimes \varphi)(x) \in \text{Fil}^1 S \otimes_{\mathfrak{S}} \mathfrak{M}\}.$$

Define the map φ_1 as the composite

$$\varphi_1 : \text{Fil}^1 M \xrightarrow{1 \otimes \varphi} \text{Fil}^1 S \otimes_{\mathfrak{S}} \mathfrak{M} \xrightarrow{\varphi_1 \otimes 1} S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} = M.$$

This defines a functor $F : \mathrm{BT}_{\mathfrak{S}}(\varphi) \rightarrow {}'(\mathrm{Mod}/S)$. Now let $\mathfrak{M} \in (\mathrm{Mod}/\mathfrak{S})$. Since \mathfrak{M} is obtained from successive extensions of free \mathfrak{S}/p -modules, $\mathrm{Tor}_{\mathfrak{S}}^1(S/\mathrm{Fil}^1 S, \mathfrak{M}) = 0$. Therefore, $\mathrm{Fil}^1 S \otimes_{\mathfrak{S}} \mathfrak{M}$ is a submodule of $S \otimes_{\mathfrak{S}} \mathfrak{M}$, and we can similarly define a functor from $(\mathrm{Mod}/\mathfrak{S})$ to $'(\mathrm{Mod}/S)$ which we also denote by F .

Proposition 2.5. (cf. [21, Proposition 1.1.11]) *The functor F on $\mathrm{BT}_{\mathfrak{S}}(\varphi)$ (resp. on $(\mathrm{Mod}/\mathfrak{S})$) takes values in $\mathrm{BT}_S(\varphi)$ (resp. in (Mod/S)). Both functors are exact and fully faithful.*

Proof. The statement for F on $(\mathrm{Mod}/\mathfrak{S})$ is a part of [21, Proposition 1.1.11].

Consider the functor $F : \mathrm{BT}_{\mathfrak{S}}(\varphi) \rightarrow {}'(\mathrm{Mod}/S)$. Let $\mathfrak{M} \in \mathrm{BT}_{\mathfrak{S}}(\varphi)$, and $M := F(\mathfrak{M}) \in {}'(\mathrm{Mod}/S)$. Since $S/\mathrm{Fil}^1 S \cong \mathcal{O}_K$ which is p -torsion free, $M/\mathrm{Fil}^1 M$ is p -torsion free. Furthermore, since $\mathrm{coker}(1 \otimes \varphi : \varphi^*(\mathfrak{M}) \rightarrow \mathfrak{M})$ is killed by $E(u)$ and $\varphi_1(E(u)) = c$ is a unit in S , $\varphi_1(\mathrm{Fil}^1 M)$ generates M over S . Thus, $F(\mathfrak{M}) \in \mathrm{BT}_S(\varphi)$.

We see that F is exact by the same argument as in the proof of [21, Proposition 1.1.11], as follows: Let

$$0 \rightarrow \mathfrak{M}' \rightarrow \mathfrak{M} \rightarrow \mathfrak{M}'' \rightarrow 0$$

be a short exact sequence in $\mathrm{BT}_{\mathfrak{S}}(\varphi)$, and let $M' := F(\mathfrak{M}')$, $M := F(\mathfrak{M})$, and $M'' := F(\mathfrak{M}'')$. Since \mathfrak{M}'' is free over \mathfrak{S} , we obtain an exact sequence of S -modules

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

and a left exact sequence

$$0 \rightarrow \mathrm{Fil}^1 M' \rightarrow \mathrm{Fil}^1 M \rightarrow \mathrm{Fil}^1 M''.$$

To see the surjectivity of the final map, let $x \in \mathrm{Fil}^1 M''$. Since $\mathrm{Fil}^1 S + \mathfrak{S} = S$ and

$\text{Fil}^1 S \cdot M \subset \text{Fil}^1 M$, we may assume that x is the image of some $\tilde{x} \in \varphi^*(\mathfrak{M}'')$. Since $x \in \text{Fil}^1 M''$, we have $(1 \otimes \varphi)(\tilde{x}) \in E(u)\mathfrak{M}''$. As $1 \otimes \varphi : \varphi^*(\mathfrak{M}'') \rightarrow \mathfrak{M}''$ is injective, there exists an $\tilde{y} \in \varphi^*(\mathfrak{M})$ which maps to \tilde{x} and satisfies $(1 \otimes \varphi)(\tilde{y}) \in E(u)\mathfrak{M}$. Then the image of \tilde{y} in M lies in $\text{Fil}^1 M$ and maps to x .

Lastly, note that for $\mathfrak{M} \in \text{BT}_{\mathfrak{S}}(\varphi)$ and for each $n \geq 1$, \mathfrak{M}/p^n equipped with the induced φ is in $(\text{Mod}/\mathfrak{S})$. And by Lemma 2.4, we see that $F(\mathfrak{M}/p^n) = F(\mathfrak{M})/p^n$ has the natural induced structure as an object in (Mod/S) . For another $\mathfrak{N} \in \text{BT}_{\mathfrak{S}}(\varphi)$, we have

$$\begin{aligned} \text{Hom}_{'(\text{Mod}/S)}(F(\mathfrak{N}), F(\mathfrak{M})) &= \varprojlim_n \text{Hom}_{'(\text{Mod}/S)}(F(\mathfrak{N}), F(\mathfrak{M})/p^n) \\ &= \varprojlim_n \text{Hom}_{'(\text{Mod}/S)}(F(\mathfrak{N})/p^n, F(\mathfrak{M})/p^n) \\ &= \varprojlim_n \text{Hom}_{'(\text{Mod}/\mathfrak{S})}(\mathfrak{N}/p^n, \mathfrak{M}/p^n) \\ &= \text{Hom}_{'(\text{Mod}/\mathfrak{S})}(\mathfrak{N}, \mathfrak{M}). \end{aligned}$$

since the functor on $(\text{Mod}/\mathfrak{S})$ is fully faithful. Thus, F is fully faithful. \square

2.3 Equivalence of Categories

In this section, we study the equivalence of categories introduced in the previous section, and prove Theorem 1.2. The following proposition is shown in [7].

Proposition 2.6. ([7, Theorem 2.2.1]) *The functor $F : \text{BT}_{\mathfrak{S}}(\varphi) \rightarrow \text{BT}_S(\varphi)$ is an equivalence of categories.*

Proof. This is [7, Theorem 2.2.1]. We sketch the proof here. Since F is fully faithful by Proposition 2.5, it remains to show that F is essentially surjective. Let $M \in \text{BT}_S(\varphi)$, and denote by $M_d(\mathfrak{S})$ the set of $d \times d$ matrices with coefficients in \mathfrak{S} . First, we construct $(\alpha_1^{(n)}, \dots, \alpha_d^{(n)}) \in \text{Fil}^1 M$ inductively on n such that:

- $(e_1^{(n)}, \dots, e_d^{(n)}) := c^{-1}\varphi_1(\alpha_1^{(n)}, \dots, \alpha_d^{(n)})$ is a basis of M , and
- there exist matrices $B^{(n)} \in M_d(\mathfrak{S})$ and $C^{(n)} \in M_d(p^n \text{Fil}^{n+p} S)$ such that

$$(\alpha_1^{(n)}, \dots, \alpha_d^{(n)}) = (e_1^{(n)}, \dots, e_d^{(n)})(B^{(n)} + C^{(n)}).$$

The construction for $n = 0$ is trivial, and for the inductive step, we set

$$(\alpha_1^{(n+1)}, \dots, \alpha_d^{(n+1)}) = (e_1^{(n)}, \dots, e_d^{(n)})B^{(n)}.$$

Let α_i (resp. e_i , B) be the limit of $\alpha_i^{(n)}$ (resp. $e_i^{(n)}$, $B^{(n)}$) as $n \rightarrow \infty$. Then $e_i = \frac{1}{c}\varphi_1(\alpha_i)$, and (e_1, \dots, e_d) is a basis of M . We have $(\alpha_1, \dots, \alpha_d) = (e_1, \dots, e_d)B$ and $B \in M_d(\mathfrak{S})$, $\alpha_i \in \text{Fil}^1 M$. Furthermore, since $\text{Fil}^1 M = \sum_{i=1}^d S\alpha_i^{(0)} + \text{Fil}^p S \cdot M$, we have

$$\text{Fil}^1 M = \sum_{i=1}^d S\alpha_i + \text{Fil}^p S \cdot M. \quad (2.1)$$

From equation (2.1), it is easy to see that there exists $\tilde{B} \in M_d(\mathfrak{S})$ such that $B\tilde{B} = E(u)I$, where I is the identity matrix. We set $\mathfrak{M} \in \text{BT}_{\mathfrak{S}}(\varphi)$ having a basis (f_1, \dots, f_d) and endowed with φ such that $\varphi(f_1, \dots, f_d) = (f_1, \dots, f_d)\tilde{B}$. Then $F(\mathfrak{M}) = M$. \square

Let $D^b(\text{BT}_S(\varphi))$ denote the bounded derived category of the exact category $\text{BT}_S(\varphi)$. Let $(\text{Mod}/S)^{\bullet}$ denote the full subcategory of $D^b(\text{BT}_S(\varphi))$ consisting of objects represented by two-term complexes $M^{\bullet} = M_1 \rightarrow M_2$ in degrees -1 and 0 , such that $H^{-1}(M^{\bullet}) = 0$ and $H^0(M^{\bullet})$ is annihilated by some power of p . More concretely, $(\text{Mod}/S)^{\bullet}$ is the category obtained by taking the category of two-term complexes M^{\bullet} as above, dividing by homotopies, and inverting quasi-isomorphisms. Similarly, we denote by $D^b(p\text{-div}/\mathcal{O}_K)$ the

bounded derived category of the category of p -divisible groups over \mathcal{O}_K , and let $(p\text{-Gr}/\mathcal{O}_K)$ be the full subcategory consisting of objects represented by isogenies of p -divisible groups $G_1 \rightarrow G_2$ in degrees -1 and 0 .

We show that the assignment $M^\bullet \mapsto H^0(M^\bullet)$ defines a functor from $(\text{Mod}/S)^\bullet$ to (Mod/S) .

Lemma 2.7. *Let $\mathfrak{M}_1, \mathfrak{M}_2 \in \text{BT}_{\mathfrak{S}}(\varphi)$ having the same rank over \mathfrak{S} . Suppose we have a monomorphism $f : \mathfrak{M}_1 \rightarrow \mathfrak{M}_2$ in $\text{BT}_{\mathfrak{S}}(\varphi)$, and let $\mathfrak{M} = \text{coker}(f)$. Then \mathfrak{M} is killed by some power of p .*

Proof. Note first that $\mathfrak{M} \in {}'(\text{Mod}/\mathfrak{S})$. Denote by $U \subset \text{Spec}(\mathfrak{S}[\frac{1}{p}])$ the largest open subscheme over which $\mathfrak{M} = 0$, and let Z be its reduced complement. Since f is injective, $\det(f) \neq 0$ and Z is the reduced subscheme corresponding to $V(\det(f))$. In particular, $Z = V(g)$ for some $g \in \mathfrak{S}[\frac{1}{p}]$.

Suppose Z is non-empty. Note that the map $1 \otimes \varphi : \varphi^*(\mathfrak{M}) \rightarrow \mathfrak{M}$ becomes an isomorphism after inverting $E(u)$. Thus, the roots of g (i.e. the elements $x \in \bar{K}$ such that its p -adic norm $|x| < 1$ and $g(x) = 0$) are contained in those of $\varphi(g)E(u)$, and the roots of $\varphi(g)$ are contained in those of $gE(u)$. Suppose that g has a non-zero root. Let x be a non-zero root whose p -adic norm is the smallest, and let y be a root whose norm is the largest. Then $\varphi(g)$ has a root w with $|w| = |y^{\frac{1}{p}}| > |y|$, and every non-zero root of $\varphi(g)$ has norm at least $|x^{\frac{1}{p}}| > |x|$. Thus, w and x are roots of $E(u)$, so that $|x| = |\varpi| = |w| > |y|$, which is a contradiction. So g does not have any non-zero root, and $Z = V(u)$.

Therefore, we have $\det(f)\mathfrak{S}[\frac{1}{p}] = u^i\mathfrak{S}[\frac{1}{p}]$ for some $i \geq 0$. Denote $K_0 := W(k)[\frac{1}{p}]$, and let $J := u^i K_0[[u]]$ be the ideal of $K_0[[u]]$. Since the map $1 \otimes \varphi : \varphi^*(\mathfrak{M}) \rightarrow \mathfrak{M}$ becomes an isomorphism after tensoring with $K_0[[u]]$, we have $\varphi(J)K_0[[u]] = J$, i.e., $u^{pi} = u^i$. Thus, $i = 0$, and Z must be empty. \square

Let $M^\bullet = M_1 \rightarrow M_2 \in (\text{Mod}/S)^\bullet$. By Proposition 2.6, there exist $\mathfrak{M}_1, \mathfrak{M}_2 \in \text{BT}_{\mathfrak{S}}(\varphi)$ with a morphism $f : \mathfrak{M}_1 \rightarrow \mathfrak{M}_2$ in $\text{BT}_{\mathfrak{S}}(\varphi)$ which correspond to M^\bullet via the functor F . Note that f is injective since $F(f)$ is injective. Let $\mathfrak{M} = \text{coker}(f) \in {}'(\text{Mod}/\mathfrak{S})$. By Lemma 2.7, \mathfrak{M} is killed by a power of p . Since \mathfrak{S} is a regular ring of dimension 2 and \mathfrak{M} has \mathfrak{S} -projective dimension 1, the Auslander-Buchsbaum theorem implies that \mathfrak{M} has no section supported on the closed point of \mathfrak{S} . Thus, for non-negative integers i , the quotients $\mathfrak{M}[p^{i+1}]/\mathfrak{M}[p^i]$ are finite free over \mathfrak{S}/p . Furthermore, by descending induction on i , we see that

$$1 \otimes \varphi : \varphi^*(\mathfrak{M}[p^{i+1}]/\mathfrak{M}[p^i]) \rightarrow \mathfrak{M}[p^{i+1}]/\mathfrak{M}[p^i]$$

is injective and its cokernel is killed by $E(u)$. So $\mathfrak{M}[p^{i+1}]/\mathfrak{M}[p^i] \in (\text{Mod}/\mathfrak{S})$ for each i , and therefore $\mathfrak{M} \in (\text{Mod}/\mathfrak{S})$. Since S is p -torsion free and \mathfrak{M} is obtained by a successive extension of free \mathfrak{S}/p -modules, we have $\text{Tor}_{\mathfrak{S}}^1(S, \mathfrak{M}) = 0$ where we consider S as a \mathfrak{S} -module via φ . Note further that $1 \otimes \varphi : \varphi^*(\mathfrak{M}) \rightarrow \mathfrak{M}$ is injective by [21, Lemma 1.1.9]. Thus, we can apply the same reasoning as in the proof of Proposition 2.5 to see that the sequence

$$0 \rightarrow F(\mathfrak{M}_1) \rightarrow F(\mathfrak{M}_2) \rightarrow F(\mathfrak{M}) \rightarrow 0$$

is exact in $'(\text{Mod}/S)$, so $H^0(M) = F(\mathfrak{M}) \in (\text{Mod}/S)$. This shows that the assignment $M^\bullet \mapsto H^0(M^\bullet)$ defines a functor from $(\text{Mod}/S)^\bullet$ to (Mod/S) .

Proposition 2.8. *The functor $M^\bullet \mapsto H^0(M^\bullet)$ induces an equivalence between $(\text{Mod}/S)^\bullet$ and (Mod/S) .*

Proof. Consider two objects $M^\bullet = M_1 \rightarrow M_2$ and $N^\bullet = N_1 \rightarrow N_2 \in (\text{Mod}/S)^\bullet$. Suppose we have a morphism $f : H^0(M^\bullet) \rightarrow H^0(N^\bullet)$ in (Mod/S) . Let $P_2 = M_2 \oplus N_2 \in \text{BT}_S(\varphi)$ with the natural induced Frobenius and filtration structure. We have the natural surjection

$P_2 \twoheadrightarrow H^0(N^\bullet)$ given as the sum of the composite $f \circ (M_2 \rightarrow H^0(M^\bullet))$ and $N_2 \rightarrow H^0(N^\bullet)$. Let $P_1 := \ker(P_2 \rightarrow H^0(N^\bullet))$ equipped with the Frobenius and filtration structure induced from P_2 . Since $H^0(N^\bullet)$ has S -projective dimension 1 and P_2 is finite free over S , P_1 is finite free S -module. $P_1/\text{Fil}^1 P_1$ is p -torsion free as $P_2/\text{Fil}^1 P_2$ is p -torsion free. Since $\varphi_1 : \text{Fil}^1(H^0(N^\bullet)) \rightarrow H^0(N^\bullet)$ is injective, $\varphi_1(\text{Fil}^1 P_1)$ generates P_1 over S . Thus, $P_1 \in \text{BT}_S(\varphi)$. Denote $P^\bullet = P_1 \rightarrow P_2 \in (\text{Mod}/S)^\bullet$. We have natural morphisms $M^\bullet \rightarrow P^\bullet$ inducing $f : H^0(M^\bullet) \rightarrow H^0(P^\bullet) = H^0(N^\bullet)$, and $N^\bullet \rightarrow P^\bullet$ inducing $H^0(N^\bullet) = H^0(P^\bullet)$. Thus, it follows from the definition of the category $(\text{Mod}/S)^\bullet$ that the functor is fully faithful.

We now show the essential surjectivity. Let $M \in (\text{Mod}/S)$. By Corollary 2.3, there exist $e_1, \dots, e_d \in M$ and $d_1 \in \{1, \dots, d\}$ such that

$$M = \sum_{i=1}^d S e_i, \quad \text{and} \quad \text{Fil}^1 M = \sum_{i=1}^{d_1} S e_i + \sum_{i=d_1+1}^d \text{Fil}^1 S \cdot e_i.$$

Let M_2 be the free S -module with a basis $\{f_1, \dots, f_d\}$, having the S -submodule $\text{Fil}^1 M_2$ given by

$$\text{Fil}^1 M_2 = \left(\bigoplus_{i=1}^{d_1} S f_i \right) \oplus \left(\bigoplus_{i=d_1+1}^d \text{Fil}^1 S \cdot f_i \right).$$

We define a φ -semilinear map $\varphi_1 : \text{Fil}^1 M_2 \rightarrow M_2$ as follows. By Lemma 2.1, $\varphi_1(\text{Fil}^1 M)$ generates M over S . Therefore, since S is local, there exist $a_{ij} \in S$, $1 \leq i, j \leq d$ such that the determinant of the $d \times d$ matrix $(a_{ij})_{1 \leq i, j \leq d}$ is a unit in S , and that

$$\varphi_1(e_i) = \sum_{j=1}^d a_{ij} e_j \quad \text{for } i = 1, \dots, d_1,$$

$$\varphi_1(E(u)e_i) = \sum_{j=1}^d a_{ij} e_j \quad \text{for } i = d_1 + 1, \dots, d.$$

We set

$$\begin{aligned}\varphi_1(f_i) &= \sum_{j=1}^d a_{ij} f_j \quad \text{for } i = 1, \dots, d_1, \\ \varphi_1(sf_i) &= c^{-1}\varphi_1(s) \sum_{j=1}^d a_{ij} f_j \quad \text{for } s \in \text{Fil}^1 S, i = d_1 + 1, \dots, d.\end{aligned}$$

This defines M_2 as an element of $\text{BT}_S(\varphi)$.

Define a morphism of S -modules $h : M_2 \rightarrow M$ by $h(f_i) = e_i$. Then h is compatible with the filtration and φ_1 , so it is a morphism in $'(\text{Mod}/S)$. Note that h is surjective. Let $M_1 = \ker(h : M_2 \rightarrow M)$ equipped with the Frobenius and filtration structure induced from M_2 . Then $M_1 \in \text{BT}_S(\varphi)$ by the same argument as above. For $M^\bullet = M_1 \rightarrow M_2 \in (\text{Mod}/S)^\bullet$, we have $H^0(M^\bullet) = M$. \square

Lemma 2.9. *The functor from $(p\text{-Gr}/\mathcal{O}_K)^\bullet$ to the category of finite flat group schemes over \mathcal{O}_K , given by sending an isogeny of p -divisible groups to its kernel, is an equivalence.*

Proof. Let $G^\bullet = G_1 \rightarrow G_2$ and $H^\bullet = H_1 \rightarrow H_2$ be two objects in $(p\text{-Gr}/\mathcal{O}_K)^\bullet$, and suppose we have a morphism $f : H^{-1}(G^\bullet) \rightarrow H^{-1}(H^\bullet)$ of finite flat group schemes over \mathcal{O}_K . Let $F_1 := G_1 \times H_1$, which is a p -divisible group over \mathcal{O}_K . We have a natural embedding $H^{-1}(G^\bullet) \rightarrow F_1$ given as the sum of $H^{-1}(G^\bullet) \rightarrow G_1$ and the composite $(H^{-1}(H^\bullet) \rightarrow H_1) \circ f$. The quotient $F_2 := F_1/H^{-1}(G^\bullet)$ in the category of fppf sheaves is a p -divisible group. Denote $F^\bullet = F_1 \rightarrow F_2 \in (p\text{-Gr}/\mathcal{O}_K)^\bullet$. Then we have natural morphisms $F^\bullet \rightarrow G^\bullet$ inducing $H^{-1}(F^\bullet) = H^{-1}(G^\bullet)$ and $F^\bullet \rightarrow H^\bullet$ inducing $f : H^{-1}(F^\bullet) = H^{-1}(G^\bullet) \rightarrow H^{-1}(H^\bullet)$. Thus, by the definition of the category $(p\text{-Gr}/\mathcal{O}_K)^\bullet$, the functor is fully faithful.

Furthermore, given a finite flat group scheme G over \mathcal{O}_K there exists an embedding of G into a p -divisible group G_1 over \mathcal{O}_K by [2, Theorem 3.1.1]. The quotient G_1/G in the category of fppf sheaves is a p -divisible group, so the functor is essentially surjective. \square

Lastly, we prove Breuil's classification of finite flat group schemes over \mathcal{O}_K .

Theorem 2.10. (**Breuil** [5, Theorem 4.2.1.6]) *The category of finite flat group schemes over \mathcal{O}_K is anti-equivalent to (Mod/S) .*

Proof. It is shown in [18, Proposition A.6] using Grothendieck-Messing deformation theory that there is an exact contravariant functor from the category of p -divisible groups over \mathcal{O}_K to $\text{BT}_S(\varphi)$, and that it is an anti-equivalence. Since $(p\text{-Gr}/\mathcal{O}_K)^\bullet$ is anti-equivalent to $(\text{Mod}/S)^\bullet$, the theorem follows from Proposition 2.8 and Lemma 2.9. \square

3 Potentially Semi-stable Deformation Ring

3.1 Introduction

We keep the notations as in Section 1. We study the geometry of the loci of p -adic representations of \mathcal{G}_K satisfying certain p -adic Hodge theoretic conditions, and prove Theorem 1.3 which states that being potentially semi-stable with a given p -adic Hodge-Tate type and Galois type is a Zariski-closed condition.

When the residue field k is further assumed to be finite, Kisin proved the corresponding result in [19, Theorem 2.7.6]. One of the main steps in [19] is the construction of the projective scheme which parametrizes representations of $E(u)$ -height $\leq r$ for a fixed positive integer r (cf. [19, Section 1.2]). It is obtained as a closed subscheme of the affine Grassmannian for the restriction of scalars $\mathrm{Res}_{W(k)/\mathbb{Z}_p} \mathrm{GL}_d$. But this construction does not make sense in general when k is infinite. The main difficulty is that we do not know how to analyze whether the restriction of scalars $\mathrm{Res}_{W(k)/\mathbb{Z}_p}$ for a non-affine scheme over $W(k)$ is representable by an Ind-scheme when k is infinite, even for simple examples such as $\mathbb{P}_{W(k)}^1$.

Another approach to studying the locus cut out by certain p -adic Hodge theoretic conditions, motivated by Fontaine's conjecture in [13], is to analyze torsion representations given as the subquotients of Galois stable lattices satisfying the given conditions. In this approach, we apply Schlessinger's criterion in [33] to obtain pro-representability, and avoid the above obstruction in the case k is infinite. For semi-stable (or crystalline) representations having Hodge-Tate weights in $[0, r]$, this is done by Liu in [25].

To study the refined structure of a p -adic Hodge-Tate type and Galois type of torsion representations, we use the functor given in [27] from the category of representations semi-stable over a totally ramified Galois extension K'/K to the category of lattices in filtered modules equipped with Frobenius, monodromy, and $\mathrm{Gal}(K'/K)$ -action. We generalize the

method in [24] and prove the corresponding result for the locus of semi-stable representations of a given p -adic Hodge-Tate type. Then, we study the Galois types of torsion representations and show the following theorem, which is the key to Theorem 1.3.

Theorem 3.1. *Let τ be a Galois type, and let L/K be a finite Galois extension in \bar{K} over which τ becomes trivial. Let C be a finite flat \mathcal{O}_E -algebra and $\rho : \mathcal{G}_K \rightarrow \mathrm{GL}_d(C)$ be a Galois representation such that $\rho \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is semi-stable over L having Hodge-Tate weights in $[0, r]$.*

Suppose that for each positive integer n , there exist a finite flat \mathcal{O}_E -algebra C_n , a Galois representation $\rho_n : \mathcal{G}_K \rightarrow \mathrm{GL}_d(C_n)$, and an \mathcal{O}_E -linear surjection $\beta_n : C_n \rightarrow C/p^n$ such that $C/p^n \otimes_C \rho \cong \beta'_n \circ \rho_n$ as $C[\mathcal{G}_K]$ -modules, and that $\rho_n \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is semi-stable over L having Hodge-Tate weights in $[0, r]$ and Galois type τ . Here, $\beta'_n : \mathrm{GL}_d(C_n) \twoheadrightarrow \mathrm{GL}_d(C/p^n)$ denotes the natural map induced by β_n .

Then $\rho \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ also has Galois type τ .

3.2 Torsion Representation and Construction of M_{st}

Let $K' \subset \bar{K}$ be a finite totally ramified Galois extension of K . In this section, we will first explain the construction of the functor given in [27] from the category of representations semi-stable over K' to the category of lattices in filtered modules equipped with Frobenius, monodromy, and $\mathrm{Gal}(K'/K)$ -action. Then, we will explain the result proved in [27] and [24] that one can associate a p -adic Hodge-Tate type and Galois type to a torsion representation up to some constant depending only on K' .

3.2.1 Potentially Semi-stable Representations and Filtered (φ, N, Γ) -module

For a \mathcal{G}_K -representation V over \mathbb{Q}_p , we say V is *potentially semi-stable* if there exists a finite extension L over K in \bar{K} such that V restricted to $\mathcal{G}_L := \mathrm{Gal}(\bar{K}/L)$ is semi-stable. This

means precisely that $\dim_{\mathbb{Q}_p} V = \dim_{L_0} (B_{\text{st}} \otimes_{\mathbb{Q}_p} V^\vee)^{\mathcal{G}_L}$ where L_0 is the maximal unramified subextension of L/K_0 .

Let $e' = [K' : K_0]$. We fix a uniformizer π of K' , and let $F(u)$ be the Eisenstein polynomial for π over K_0 . Denote by $\text{Rep}_{\mathbb{Q}_p}^{\text{pst}, K'}$ the category of \mathcal{G}_K -representations over \mathbb{Q}_p which become semi-stable over K' . Let $\Gamma = \text{Gal}(K'/K)$ and $\mathcal{G}_{K'} = \text{Gal}(\bar{K}/K')$.

We consider the category of filtered (φ, N, Γ) -modules whose objects are finite dimensional K_0 -vector spaces equipped with:

- a Frobenius semilinear injection $\varphi : D \rightarrow D$,
- $W(k)$ -linear map $N : D \rightarrow D$ such that $N\varphi = p\varphi N$,
- decreasing filtration $\text{Fil}^i D_{K'}$ on $D_{K'} := K' \otimes_{K_0} D$ by K' -sub-vector spaces such that $\text{Fil}^i D_{K'} = D_{K'}$ for $i \ll 0$ and $\text{Fil}^i D_{K'} = 0$ for $i \gg 0$, and
- K_0 -linear action by Γ on D which commutes with φ and N . If we extend Γ -action semilinearly to $D_{K'}$, then for any $\gamma \in \Gamma$, $\gamma(\text{Fil}^i D_{K'}) \subset \text{Fil}^i D_{K'}$.

Morphisms between filtered (φ, N, Γ) -modules are K_0 -linear maps preserving all structures. The functor $D_{\text{st}}^{K'} : V \mapsto (B_{\text{st}} \otimes_{\mathbb{Q}_p} V^\vee)^{\mathcal{G}_{K'}}$ is an equivalence between $\text{Rep}_{\mathbb{Q}_p}^{\text{pst}, K'}$ and the category of weakly admissible filtered (φ, N, Γ) -modules (cf. [8], [12]).

We define an integral structure of a filtered (φ, N, Γ) -module.

Definition 3.2. Let D be a filtered (φ, N, Γ) -module. A *lattice* M in D is a finite free $W(k)$ -submodule of D such that $M[\frac{1}{p}] := M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong D$, and $\varphi(M) \subset M$, $N(M) \subset M$, and $\gamma(M) \subset M$ for all $\gamma \in \Gamma$. For a lattice $M \subset D$, we equip $M_{K'} := \mathcal{O}_{K'} \otimes_{W(k)} M$ with a natural filtration by $\mathcal{O}_{K'}$ -submodules, given by $\text{Fil}^i M_{K'} = M_{K'} \cap \text{Fil}^i D_{K'}$. If M_1, M_2 are lattices in filtered (φ, N, Γ) -modules D_1, D_2 respectively, then a morphism $f : M_1 \rightarrow M_2$ is a $W(k)$ -linear map such that $f \otimes_{\mathbb{Z}_p} \mathbb{Q}_p : D_1 \rightarrow D_2$ is a morphism of filtered (φ, N, Γ) -modules.

Note that for a lattice M in a filtered (φ, N, Γ) -module, the graded modules $\mathrm{gr}^i M_{K'} = \mathrm{Fil}^i M_{K'} / \mathrm{Fil}^{i+1} M_{K'}$ is torsion free by the definition of the filtration.

Let r be a positive integer. Denote by $L^r(\varphi, N, \Gamma)$ the category of lattices in filtered (φ, N, Γ) -modules D satisfying $\mathrm{Fil}^0 D_{K'} = D_{K'}$ and $\mathrm{Fil}^{r+1} D_{K'} = 0$. Let $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{pst}, K', r}$ be the full subcategory of $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{pst}, K'}$ whose objects have Hodge-Tate weights in $[0, r]$, and let $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{pst}, K', r}$ be the category of \mathcal{G}_K -stable \mathbb{Z}_p -lattices of representations in $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{pst}, K', r}$. The following theorem is proved in [27]:

Theorem 3.3. (cf. [27, Theorem 2.3]) *There exists a faithful contravariant functor M_{st} from $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{pst}, K', r}$ to $L^r(\varphi, N, \Gamma)$. If we denote by $M_{\mathrm{st}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ the functor M_{st} associated to the isogeny categories, then there exists a natural isomorphism of functors between $M_{\mathrm{st}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $D_{\mathrm{st}}^{K'}$.*

3.2.2 Construction of M_{st}

Following [27], we now explain briefly the construction of the functor M_{st} given in Theorem 3.3. We first recall the definitions of period rings in p -adic Hodge theory.

Let S' be the p -adic completion of the divided power-envelope of \mathfrak{S} with respect to the ideal $(F(u))$. Denote $S'_{K_0} := S'[\frac{1}{p}]$. Let $\mathfrak{R} := \varprojlim_{x \mapsto x^p} \mathcal{O}_{\bar{K}}/p$. By the universal property of the Witt vectors $W(\mathfrak{R})$, there is a unique surjection $\theta : W(\mathfrak{R}) \rightarrow \mathcal{O}_{\hat{K}}$, which lifts the projection $\mathfrak{R} \rightarrow \mathcal{O}_{\bar{K}}/p$ onto the first factor of the inverse limit. Here, \hat{K} denotes the p -adic completion of \bar{K} , and $\mathcal{O}_{\hat{K}}$ is its ring of integers. We denote by B_{dR}^+ the $\ker(\theta)$ -adic completion of $W(\mathfrak{R})[\frac{1}{p}]$. Let A_{cris} be the p -adic completion of the divided power-envelope of $W(\mathfrak{R})$ with respect to $\ker(\theta)$. We fix a compatible system of a p^n -th root $\pi_n \in \mathcal{O}_{\bar{K}}$ of π for non-negative integers n , and let $\underline{\pi} := (\pi_n) \in \mathfrak{R}$. We have an embedding $\mathfrak{S} \hookrightarrow W(\mathfrak{R})$ mapping u to $[\underline{\pi}]$, and hence the embeddings $\mathfrak{S} \hookrightarrow S' \hookrightarrow A_{\mathrm{cris}}$ compatible with Frobenius endomorphisms. Let $B_{\mathrm{cris}}^+ = A_{\mathrm{cris}}[\frac{1}{p}]$. Let $\mathbf{u} = \log[\underline{\pi}]$, and $B_{\mathrm{st}}^+ = B_{\mathrm{cris}}^+[\mathbf{u}]$. We also fix a

compatible system of a primitive p^n -th root of unity $\zeta_{p^n} \in \mathcal{O}_{\bar{K}}$ for non-negative integers n , and let $\underline{\epsilon} := (\zeta_{p^n}) \in \mathfrak{R}$. Let $t = \log[\underline{\epsilon}] \in B_{\text{dR}}^+$. Note that we also have $t \in A_{\text{cris}}$. Let $B_{\text{dR}} = B_{\text{dR}}^+[\frac{1}{t}]$, $B_{\text{cris}} = B_{\text{cris}}^+[\frac{1}{t}]$, and $B_{\text{st}} = B_{\text{st}}^+[\frac{1}{t}]$.

We denote by $\mathcal{O}_{\mathcal{E}}$ the p -adic completion of $\mathfrak{S}[\frac{1}{u}]$, and let $\mathcal{E} = \text{Frac}(\mathcal{O}_{\mathcal{E}})$. Let $\hat{\mathcal{E}}^{\text{ur}}$ be the p -adic completion of the maximal unramified subextension of \mathcal{E} in $W(\text{Frac}(\mathfrak{R}))[\frac{1}{p}]$, and $\mathcal{O}_{\hat{\mathcal{E}}^{\text{ur}}}$ its ring of integers. We let $\mathfrak{S}^{\text{ur}} = \mathcal{O}_{\hat{\mathcal{E}}^{\text{ur}}} \cap W(\mathfrak{R})$.

We let $K'_{\infty} = \bigcup_{n=1}^{\infty} K'(\pi_n)$ and $K'_{p^{\infty}} = \bigcup_{n=1}^{\infty} K'(\zeta_{p^n})$. Let $K'_c = K'_{\infty} K'_{p^{\infty}}$, which is the Galois closure of K'_{∞} over K' . Let $\hat{\mathcal{G}} = \text{Gal}(K'_c/K')$, $\mathcal{G}_{\infty} = \text{Gal}(\bar{K}/K'_{\infty})$, and $\mathcal{H}_{K'} = \text{Gal}(K'_c/K'_{\infty})$. For each $g \in \mathcal{G}_K$, we write $\underline{\epsilon}(g) := \frac{g(\pi)}{\pi}$, which is a cocycle from \mathcal{G}_K to \mathfrak{R}^* . Write

$$t^{\{i\}} = \frac{t^i}{p^{q(i)} q(i)!}$$

where $q(i)$ is defined by $i = q(i)(p-1) + r(i)$ with $0 \leq r(i) < p-1$. We define

$$\mathcal{R}_{K_0} := \left\{ \sum_{i=0}^{\infty} a_i t^{\{i\}} \mid a_i \in S'_{K_0}, \ a_i \rightarrow 0 \text{ } p\text{-adically as } i \rightarrow \infty \right\}.$$

We have a natural map $\nu : W(\mathfrak{R}) \rightarrow W(\bar{k})$ induced by the projection $\mathfrak{R} \rightarrow \bar{k}$, which can be seen to extend uniquely to $\nu : B_{\text{cris}}^+ \rightarrow W(\bar{k})[\frac{1}{p}]$. For any subring $A \subset B_{\text{cris}}^+$, write $I_+ A := A \cap \ker(\nu)$. We have $I_+ \mathfrak{S} = u \mathfrak{S}$ and

$$I_+ S' = \left\{ \sum_{i=1}^{\infty} \frac{b_i}{[\frac{i}{e'}]!} u^i \mid b_i \in W(k), \ b_i \rightarrow 0 \text{ } p\text{-adically as } i \rightarrow \infty \right\}.$$

Define $\hat{\mathcal{R}} = W(\mathfrak{R}) \cap \mathcal{R}_{K_0}$ and $I_+ = I_+ \hat{\mathcal{R}}$. The following lemma is proved in [26].

Lemma 3.4. ([26, Lemma 2.2.1])

1. $\hat{\mathcal{R}}$ (resp. \mathcal{R}_{K_0}) is a φ -stable \mathfrak{S} -algebra as a subring in $W(\mathfrak{R})$ (resp. B_{cris}^+).

2. $\hat{\mathcal{R}}$ and I_+ (resp. \mathcal{R}_{K_0} and $I_+\mathcal{R}_{K_0}$) are $\mathcal{G}_{K'}$ -stable. The $\mathcal{G}_{K'}$ -actions on $\hat{\mathcal{R}}$ and I_+ (resp. \mathcal{R}_{K_0} and $I_+\mathcal{R}_{K_0}$) factor through $\hat{\mathcal{G}}$.
3. $\mathcal{R}_{K_0}/I_+\mathcal{R}_{K_0} \cong K_0$ and $\hat{\mathcal{R}}/I_+ \cong S'/I_+S' \cong \mathfrak{S}/u\mathfrak{S} \cong W(k)$.

Let r be a positive integer. A *Kisin module of height r* is a pair $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ where \mathfrak{M} is a finite free \mathfrak{S} -module, and $\varphi_{\mathfrak{M}} : \mathfrak{M} \rightarrow \mathfrak{M}$ is a φ -semilinear map such that the cokernel of the induced map $1 \otimes \varphi_{\mathfrak{M}} : \varphi^*(\mathfrak{M}) \rightarrow \mathfrak{M}$ is killed by $E(u)^r$. A morphism between two Kisin modules $\mathfrak{M}_1, \mathfrak{M}_2$ is a morphism as \mathfrak{S} -modules compatible with $\varphi_{\mathfrak{M}_i}$. Let $\text{Mod}_{\mathfrak{S}}^r(\varphi)$ denote the category of Kisin modules of height r . For $(\mathfrak{M}, \varphi_{\mathfrak{M}}) \in \text{Mod}_{\mathfrak{S}}^r(\varphi)$, we write $\hat{\mathfrak{M}} = \hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$. The Frobenius $\varphi_{\mathfrak{M}}$ on \mathfrak{M} naturally extends to $\hat{\mathfrak{M}}$ by $\varphi_{\hat{\mathfrak{M}}}(a \otimes m) = \varphi_{\hat{\mathcal{R}}}(a) \otimes \varphi_{\mathfrak{M}}(m)$.

Definition 3.5. A $(\varphi, \hat{\mathcal{G}})$ -module of height r is a triple $(\mathfrak{M}, \varphi, \hat{\mathcal{G}})$ satisfying the following:

- $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ is Kisin module of height r .
- $\hat{\mathcal{G}}$ acts $\hat{\mathcal{R}}$ -semilinearly on $\hat{\mathfrak{M}}$, and $\hat{\mathcal{G}}$ -action on $\hat{\mathfrak{M}}$ commutes with $\varphi_{\hat{\mathfrak{M}}}$.
- Considering \mathfrak{M} as a $\varphi(\mathfrak{S})$ -submodule of $\hat{\mathfrak{M}}$, we have $\mathfrak{M} \subset \hat{\mathfrak{M}}^{\mathcal{H}_{K'}}$.
- $\hat{\mathcal{G}}$ acts trivially on $\hat{\mathfrak{M}}/I_+\hat{\mathfrak{M}}$.

A morphism between two $(\varphi, \hat{\mathcal{G}})$ -modules $\mathfrak{M}_1, \mathfrak{M}_2$ of height r is a morphism in $\text{Mod}_{\mathfrak{S}}^r(\varphi)$ which commutes with $\hat{\mathcal{G}}$ -action. We denote by $\text{Mod}_{\mathfrak{S}}^r(\varphi, \hat{\mathcal{G}})$ the category of $(\varphi, \hat{\mathcal{G}})$ -modules of height r . For $\hat{\mathfrak{M}} \in \text{Mod}_{\mathfrak{S}}^r(\varphi, \hat{\mathcal{G}})$, we associate a $\mathbb{Z}_p[\mathcal{G}_{K'}]$ -module $\hat{T}^{\vee}(\hat{\mathfrak{M}}) := \text{Hom}_{\hat{\mathcal{R}}, \varphi}(\hat{\mathfrak{M}}, W(\mathfrak{R}))$ with $\mathcal{G}_{K'}$ -action given by $g(f)(x) = g(f(g^{-1}(x)))$ for $g \in \mathcal{G}_{K'}$, $f \in \hat{T}^{\vee}(\hat{\mathfrak{M}})$. Here, $\mathcal{G}_{K'}$ -action on $\hat{\mathfrak{M}}$ is the natural one given by $\hat{\mathcal{G}}$ -action on $\hat{\mathfrak{M}}$. Moreover, for $\mathfrak{M} \in \text{Mod}_{\mathfrak{S}}^r(\varphi)$, we associate a $\mathbb{Z}_p[\mathcal{G}_{\infty}]$ -module $T_{\mathfrak{S}}^{\vee}(\mathfrak{M}) := \text{Hom}_{\mathfrak{S}, \varphi}(\mathfrak{M}, \mathfrak{S}^{\text{ur}})$ similarly. The main result proved in [26] is the following.

Theorem 3.6. (cf. [26, Theorem 2.3.1, Proposition 3.1.3])

1. \hat{T}^\vee induces an anti-equivalence between $\text{Mod}_{\mathfrak{S}}^r(\varphi, \hat{\mathcal{G}})$ and the category of $\mathcal{G}_{K'}$ -stable \mathbb{Z}_p -lattices in semi-stable representations of $\mathcal{G}_{K'}$ having Hodge-Tate weights in $[0, r]$.
2. \hat{T}^\vee induces a natural $W(\mathfrak{R})$ -linear injection

$$\hat{\iota} : W(\mathfrak{R}) \otimes_{\mathcal{A}} \hat{\mathfrak{M}} \rightarrow W(\mathfrak{R}) \otimes_{\mathbb{Z}_p} \hat{T}(\hat{\mathfrak{M}})$$

such that $\hat{\iota}$ is compatible with Frobenius maps and $\mathcal{G}_{K'}$ -actions on both sides. Here, $\hat{T}(\hat{\mathfrak{M}}) := \text{Hom}_{\mathbb{Z}_p}(\hat{T}^\vee(\hat{\mathfrak{M}}), \mathbb{Z}_p)$.

3. There exists a natural isomorphism $T_{\mathfrak{S}}^\vee(\mathfrak{M}) \xrightarrow{\cong} \hat{T}^\vee(\hat{\mathfrak{M}})$ of $\mathbb{Z}_p[\mathcal{G}_\infty]$ -modules.

To construct the functor M_{st} , we need to establish a connection between $(\varphi, \hat{\mathcal{G}})$ -modules and filtered (φ, N) -modules. Let $V \in \text{Rep}_{\mathbb{Q}_p}^{\text{pst}, K', r}$, and let $T \subset V$ be a \mathcal{G}_K -stable \mathbb{Z}_p -lattice. By Theorem 3.6, there exists a unique $\mathfrak{M} \in \text{Mod}_{\mathfrak{S}}^r(\varphi, \hat{\mathcal{G}})$ such that $\hat{T}^\vee(\hat{\mathfrak{M}}) = T$ as $\mathbb{Z}_p[\mathcal{G}_{K'}]$ -modules. Let $\mathcal{D} := S'_{K_0} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ equipped with the Frobenius endomorphism given by $\varphi_{\mathcal{D}} = \varphi_{S'_{K_0}} \otimes \varphi_{\mathfrak{M}}$. Let $D = \mathcal{D}/(I_+ S_{K_0})\mathcal{D}$, which is a finite K_0 -vector space equipped with the Frobenius induced from $\varphi_{\mathcal{D}}$. By [3, Proposition 6.2.1.1], there exists a unique section $s : D \rightarrow \mathcal{D}$ compatible with the Frobenius morphisms on both sides. Thus, $\mathcal{D} = S'_{K_0} \otimes_{K_0} D$ if we identify D with $s(D)$. Since $B_{\text{cris}}^+ \otimes_{\hat{\mathcal{G}}} \hat{\mathfrak{M}} \cong B_{\text{cris}}^+ \otimes_{K_0} D$, the map $\hat{\iota}$ given in Theorem 3.6(2) induces a natural injection $D \hookrightarrow B_{\text{cris}}^+ \otimes_{\mathbb{Z}_p} T$.

On the other hand, the functor $D_{\text{st}}^{K'}$ induces an injection

$$\iota : B_{\text{st}}^+ \otimes_{K_0} D_{\text{st}}^{K'}(V) \rightarrow B_{\text{st}}^+ \otimes_{\mathbb{Q}_p} V^\vee$$

such that ι is compatible with φ , N , filtration, and $\mathcal{G}_{K'}$ -action on both sides. The following

is proved in [27].

Proposition 3.7. (cf. [27, Proposition 2.6, Corollary 2.7, 2.8]) *There exists a unique K_0 -linear isomorphism $i : D_{\text{st}}^{K'}(V) \rightarrow D$ such that i is compatible with the Frobenius morphisms on both sides and makes the following diagram commutative:*

$$\begin{array}{ccc} D_{\text{st}}^{K'}(V) & \hookrightarrow & T \otimes_{\mathbb{Z}_p} B_{\text{st}}^+ \\ \downarrow i & & \downarrow \text{mod } u \\ D & \hookrightarrow & T \otimes_{\mathbb{Z}_p} B_{\text{cris}}^+ \end{array}$$

Furthermore, such i is functorial.

Note that

$$\mathfrak{M}/u\mathfrak{M} \cong \varphi^*(\mathfrak{M})/u\varphi^*(\mathfrak{M}) \subset \mathscr{D}/I_+ S'_{K_0} \mathscr{D} = D.$$

We set $M_{\text{st}}(T) \subset D_{\text{st}}^{K'}(V)$ to be the inverse image of $\varphi^*(\mathfrak{M})/u\varphi^*(\mathfrak{M})$ under the isomorphism $i : D_{\text{st}}^{K'}(V) \rightarrow D$ given in Proposition 3.7. $M_{\text{st}}(T)$ is a finite free $W(k)$ -lattice in $D_{\text{st}}^{K'}(V)$ stable under Frobenius. Furthermore, it is proved in [27, Corollary 2.12, Proposition 2.15] that $M_{\text{st}}(T)$ is stable under \mathcal{G}_K -action and N on $D_{\text{st}}^{K'}(V)$. Thus, $M_{\text{st}}(T)$ is a lattice of the filtered (φ, N, Γ) -module $D_{\text{st}}^{K'}(V)$.

3.2.3 Potentially Semi-stable Torsion Representations

We now associate torsion filtered (φ, N, Γ) -modules to potentially semi-stable torsion representations. Denote by $\text{Rep}_{\text{tor}}^{\text{pst}, K', r}$ the category of torsion representations L semi-stable over K' and of height r , in a sense that there exist lattices $\mathcal{L}_1, \mathcal{L}_2 \in \text{Rep}_{\mathbb{Z}_p}^{\text{pst}, K', r}$ with a \mathcal{G}_K -equivariant injection $j : \mathcal{L}_1 \hookrightarrow \mathcal{L}_2$ such that $L \cong \mathcal{L}_2/j(\mathcal{L}_1)$ as $\mathbb{Z}_p[\mathcal{G}_K]$ -modules, and L is killed by some power of p . Morphisms between two torsion representations in $\text{Rep}_{\text{tor}}^{\text{pst}, K', r}$ are morphisms of $\mathbb{Z}_p[\mathcal{G}_K]$ -modules. We call such $(\mathcal{L}_1, \mathcal{L}_2, j)$ a *lift* of L . We will some-

times denote simply by j a lift of L . Note that a lift of $L \in \text{Rep}_{\text{tor}}^{\text{pst}, K', r}$ is not unique. Let $L, L' \in \text{Rep}_{\text{tor}}^{\text{pst}, K', r}$ with lifts $(\mathcal{L}_1, \mathcal{L}_2, j), (\mathcal{L}'_1, \mathcal{L}'_2, j')$ respectively. If $f : L \rightarrow L'$ is a morphism in $\text{Rep}_{\text{tor}}^{\text{pst}, K', r}$, we say a morphism $\tilde{f} : \mathcal{L}_2 \rightarrow \mathcal{L}'_2$ in $\text{Rep}_{\mathbb{Z}_p}^{\text{pst}, K', r}$ is a *lift* of f if $\tilde{f}(j(\mathcal{L}_1)) \subset j'(\mathcal{L}'_1)$ and \tilde{f} induces f .

We denote by $M_{\text{tor}}^{\text{fil}, r}(\varphi, N, \Gamma)$ the category whose objects are finite $W(k)$ -modules killed by some power of p and endowed with the following structures:

- a Frobenius semilinear morphism $\varphi : M \rightarrow M$,
- $W(k)$ -linear map $N : M \rightarrow M$ satisfying $N\varphi = p\varphi N$,
- $W(k)$ -linear Γ -action on M which commutes with φ and N , and
- $M_{K'} := \mathcal{O}_{K'} \otimes_{W(k)} M$ has decreasing filtration by $\mathcal{O}_{K'}$ -submodules such that $\text{Fil}^0 M_{K'} = M_{K'}$ and $\text{Fil}^{r+1} M_{K'} = 0$. Also, $\gamma(\text{Fil}^i M_{K'}) \subset \text{Fil}^i M_{K'}$ for any $\gamma \in \Gamma$.

Morphisms in $M_{\text{tor}}^{\text{fil}, r}(\varphi, N, \Gamma)$ are morphisms of $W(k)$ -modules compatible with above structures. For $L \in \text{Rep}_{\text{tor}}^{\text{pst}, K', r}$ with a lift $j : \mathcal{L}_1 \hookrightarrow \mathcal{L}_2$, we can associate an object $M_{\text{st}, j}(L) \in M_{\text{tor}}^{\text{fil}, r}(\varphi, N, \Gamma)$ as follows. By Theorem 3.3, we have the morphism $M_{\text{st}}(j) : M_{\text{st}}(\mathcal{L}_2) \rightarrow M_{\text{st}}(\mathcal{L}_1)$ in $L^r(\varphi, N, \Gamma)$ corresponding to j , and $M_{\text{st}}(j)$ is injective by [27, Corollary 3.8]. We set $M_{\text{st}, j}(L) = M_{\text{st}}(\mathcal{L}_2)/M_{\text{st}}(j)(M_{\text{st}}(\mathcal{L}_1))$. Then $M_{\text{st}, j}(L)$ has natural endomorphisms φ and N , and Γ -action induced from $M_{\text{st}}(\mathcal{L}_1)$. Furthermore, tensoring by $\mathcal{O}_{K'}$ on $M_{\text{st}}(j)$ gives the following exact sequence:

$$0 \rightarrow \mathcal{O}_{K'} \otimes_{W(k)} M_{\text{st}}(\mathcal{L}_2) \rightarrow \mathcal{O}_{K'} \otimes_{W(k)} M_{\text{st}}(\mathcal{L}_1) \xrightarrow{q} \mathcal{O}_{K'} \otimes_{W(k)} M_{\text{st}, j}(L) \rightarrow 0.$$

We define the filtration on $M_{\text{st}, j}(L)_{K'}$ by $\text{Fil}^i M_{\text{st}, j}(L)_{K'} := q(\text{Fil}^i M_{\text{st}}(\mathcal{L}_1)_{K'})$. This gives $M_{\text{st}, j}(L)$ a structure as an object in $M_{\text{tor}}^{\text{fil}, r}(\varphi, N, \Gamma)$. By the snake lemma, we further have

the following exact sequence of the associated graded modules:

$$0 \rightarrow \mathrm{gr}^i(M_{\mathrm{st}}(\mathcal{L}_2)_{K'}) \rightarrow \mathrm{gr}^i(M_{\mathrm{st}}(\mathcal{L}_1)_{K'}) \rightarrow \mathrm{gr}^i(M_{\mathrm{st},j}(L)_{K'}) \rightarrow 0.$$

If $f : L \rightarrow L'$ is a morphism in $\mathrm{Rep}_{\mathrm{tor}}^{\mathrm{pst},K',r}$ with a lift $\tilde{f} : (\mathcal{L}_1, \mathcal{L}_2, j) \rightarrow (\mathcal{L}'_1, \mathcal{L}'_2, j')$, then it induces a morphism $M_{\mathrm{st},\tilde{f}}(f) : M_{\mathrm{st},j'}(L') \rightarrow M_{\mathrm{st},j}(L)$ in $M_{\mathrm{tor}}^{\mathrm{fil},r}(\varphi, N, \Gamma)$.

Note that the above construction depends on the choice of the lift of L . However, the following theorem, which can be deduced directly from [24] and [27], shows that the construction depends on lifts only up to a constant.

Theorem 3.8. *There exists a constant c depending only on $F(u)$ and r such that the following statement holds: for any morphism $f : L \rightarrow L'$ in $\mathrm{Rep}_{\mathrm{tor}}^{\mathrm{pst},K',r}$ with lifts j, j' of L, L' respectively, there exists a morphism $\tilde{h} : M_{\mathrm{st},j'}(L') \rightarrow M_{\mathrm{st},j}(L)$ in $M_{\mathrm{tor}}^{\mathrm{fil},r}(\varphi, N, \Gamma)$ such that*

- *if there exists a morphism of lifts $\tilde{f} : j \rightarrow j'$ which lifts f , then $\tilde{h} = p^c M_{\mathrm{st},\tilde{f}}(f)$,*
- *let $f' : L' \rightarrow L''$ be a morphism in $\mathrm{Rep}_{\mathrm{tor}}^{\mathrm{pst},K',r}$, j'' a lift of L'' , and $\tilde{h}' : M_{\mathrm{st},j''}(L'') \rightarrow M_{\mathrm{st},j'}(L')$ the morphism in $M_{\mathrm{tor}}^{\mathrm{fil},r}(\varphi, N, \Gamma)$ associated to f', j' , and j'' . If there exists a morphism of lifts $\tilde{g} : j \rightarrow j''$ which lifts $f' \circ f$, then $\tilde{h} \circ \tilde{h}' = p^{2c} M_{\mathrm{st},\tilde{g}}(f' \circ f)$.*

Proof. It follows directly from [27, Theorem 3.1] and [24, Theorem 2.1.3]. \square

The following corollary is immediate.

Corollary 3.9. (cf. [27, Corollary 3.2], [24, Corollary 2.1.4]) *With notations as in Theorem 3.8, assume that $f : L \rightarrow L'$ is an isomorphism with the inverse $f^{-1} : L' \rightarrow L$. Let $\tilde{h}_1 : M_{\mathrm{st},j}(L) \rightarrow M_{\mathrm{st},j'}(L')$ be the morphism as in Theorem 3.8 associated to f^{-1}, j , and j' . Then $\tilde{h} \circ \tilde{h}_1 = p^{2c} \mathrm{Id}$ on $M_{\mathrm{st},j}(L)$ and $\tilde{h}_1 \circ \tilde{h} = p^{2c} \mathrm{Id}$ on $M_{\mathrm{st},j'}(L')$. Furthermore, the similar statement holds for the induced morphisms on $\mathrm{gr}^i(M_{\mathrm{st},j}(L)_{K'})$ and $\mathrm{gr}^i(M_{\mathrm{st},j'}(L')_{K'})$.*

3.2.4 Representation with Coefficient

Let A be a \mathbb{Z}_p -algebra, and denote by $\text{Rep}_A^{\text{pst}, K', r}$ the subcategory of $\text{Rep}_{\mathbb{Z}_p}^{\text{pst}, K', r}$ whose objects are A -modules such that \mathcal{G}_K -actions are A -linear. Morphisms in $\text{Rep}_A^{\text{pst}, K', r}$ are morphisms of $A[\mathcal{G}_K]$ -modules. Let $\text{Rep}_{\text{tor}, A}^{\text{pst}, K', r}$ be the subcategory of $\text{Rep}_{\text{tor}}^{\text{pst}, K', r}$ whose objects have lifts in $\text{Rep}_A^{\text{pst}, K', r}$, and the morphisms in $\text{Rep}_{\text{tor}, A}^{\text{pst}, K', r}$ are morphisms of $A[\mathcal{G}_K]$ -modules. For $L \in \text{Rep}_{\text{tor}, A}^{\text{pst}, K', r}$ having a lift $j : \mathcal{L}_1 \hookrightarrow \mathcal{L}_2$ in $\text{Rep}_A^{\text{pst}, K', r}$, note that $M_{\text{st}}(\mathcal{L}_1)$ and $M_{\text{st}}(\mathcal{L}_2)$ are naturally $A \otimes_{\mathbb{Z}_p} W(k)$ -modules, and thus so is $M_{\text{st}, j}(L)$.

Proposition 3.10. *Let $f : L \rightarrow L'$ be a morphism in $\text{Rep}_{\text{tor}, A}^{\text{pst}, K', r}$, and let j and j' be lifts in $\text{Rep}_A^{\text{pst}, K', r}$ of L and L' respectively. Then, the associated morphism $\tilde{h} : M_{\text{st}, j'}(L') \rightarrow M_{\text{st}, j}(L)$ in $M_{\text{tor}}^{\text{fil}, r}(\varphi, N, \Gamma)$ as in Theorem 3.8 is a morphism of $A \otimes_{\mathbb{Z}_p} W(k)$ -modules.*

Proof. It follows immediately from [27, Proposition 3.13] and [24, Lemma 4.2.4]. \square

3.3 p -adic Hodge-Tate Type and Galois Type

3.3.1 p -adic Hodge-Tate Type

Let E be a finite extension of \mathbb{Q}_p , and let B be a finite E -algebra. Let V_B be a finite free B -module of rank d equipped with \mathcal{G}_K -action. Suppose that as a representation of \mathcal{G}_K , V_B is semi-stable over K' , i.e., $V_B \in \text{Rep}_{\mathbb{Q}_p}^{\text{pst}, K'}$. Then V_B is de Rham over K , and we set $D_{\text{dR}}^K(V_B) = (B_{\text{dR}} \otimes_{\mathbb{Q}_p} V_B^\vee)^{\mathcal{G}_K}$. For any E -algebra A , we write $A_K := A \otimes_{\mathbb{Q}_p} K$.

Lemma 3.11. (cf. [24, Lemma 4.1.2])

1. *Let B' be a finite B -algebra, and write $V_{B'} = B' \otimes_B V_B$. Then*

$$D_{\text{dR}}^K(V_{B'}) \cong B' \otimes_B D_{\text{dR}}^K(V_B), \text{ and } \text{gr}^i(D_{\text{dR}}^{K'}(V_{B'})) \cong B' \otimes_B \text{gr}^i(D_{\text{dR}}^K(V_B)).$$

2. *$D_{\text{dR}}^K(V_B)$ is a finite free B_K -module.*

Proof. (1) is proved in [24, Lemma 4.1.2]. For (2), since $D_{\text{dR}}^K(V_B) = K \otimes_{K_0} D_{\text{st}}^{K'}(V_B)$, it suffices to prove that $D_{\text{st}}^{K'}(V_B)$ is a finite free $B \otimes_{\mathbb{Q}_p} K_0$ -module. For any finite B -algebra B' , we can show similarly as in (1) that $D_{\text{st}}^{K'}(V_{B'}) \cong B' \otimes_B D_{\text{st}}^{K'}(V_B)$. Let $B_{\text{red}} = B/\mathcal{N}(B)$ where $\mathcal{N}(B)$ denotes the nilpotent ideal of B . B_{red} is a reduced Artinian ring, so there exists a ring isomorphism $B_{\text{red}} \cong \prod_{j=1}^m E_j$ for some field E_j finite over E . $E_j \otimes_{\mathbb{Q}_p} K_0$ is isomorphic to a finite direct product of fields, so $D_{\text{st}}^{K'}(V_{E_j}) \cong E_j \otimes_B D_{\text{st}}^{K'}(V_B)$ is finite projective as an $E_j \otimes_{\mathbb{Q}_p} K_0$ -module. Note that the Frobenius morphism on K_0 extends E_j -linearly to $E_j \otimes_{\mathbb{Q}_p} K_0$, and the extended Frobenius permutes the maximal ideals of $E_j \otimes_{\mathbb{Q}_p} K_0$ transitively. Therefore, $D_{\text{st}}^{K'}(V_{E_j})$ is a free $E_j \otimes_{\mathbb{Q}_p} K_0$ -module of rank d , and $D_{\text{st}}^{K'}(V_{B_{\text{red}}}) = B_{\text{red}} \otimes_B D_{\text{st}}^{K'}(V_B)$ is a free $B_{\text{red}} \otimes_{\mathbb{Q}_p} K_0$ -module of rank d .

Let $\{e_1, \dots, e_d\}$ be a $B_{\text{red}} \otimes_{\mathbb{Q}_p} K_0$ -basis of $D_{\text{st}}^{K'}(V_{B_{\text{red}}})$, and choose a lift $\hat{e}_i \in D_{\text{st}}^{K'}(V_B)$ of e_i . By Nakayama's lemma, $\{\hat{e}_1, \dots, \hat{e}_d\}$ generate $D_{\text{st}}^{K'}(V_B)$ as a $B \otimes_{\mathbb{Q}_p} K_0$ -module. Thus, we have a surjection of $B \otimes_{\mathbb{Q}_p} K_0$ -modules

$$f : \bigoplus_{i=1}^d B \otimes_{\mathbb{Q}_p} K_0 \cdot \hat{e}_i \twoheadrightarrow D_{\text{st}}^{K'}(V_B).$$

As a K_0 -vector space, $\dim_{K_0} D_{\text{st}}^{K'}(V_B) = d \cdot \dim_{\mathbb{Q}_p} B$. Thus, f is an isomorphism, and $D_{\text{st}}^{K'}(V_B)$ is a finite free $B \otimes_{\mathbb{Q}_p} K_0$ -module of rank d . \square

Let D_E be a finite E -vector space such that $D_{E,K} := D_E \otimes_E K$ is equipped with a decreasing filtration $\text{Fil}^i D_{E,K}$ of $E \otimes_{\mathbb{Q}_p} K$ -modules and $\{i \mid \text{gr}^i D_{E,K} \neq 0\} \subset \{0, \dots, r\}$. We denote $\mathbf{v} = (D_{E,K}, \{\text{Fil}^i D_{E,K}\}_{i=0, \dots, r})$. We say that V_B has *p-adic Hodge-Tate type* \mathbf{v} if $\text{gr}^i D_{\text{dR}}^K(V_B) \cong B \otimes_E \text{gr}^i D_{E,K}$ as B_K -modules for all i .

Lemma 3.12. *For a finite B -algebra B' , $V_{B'}$ has p-adic Hodge-Tate type \mathbf{v} if V_B has p-adic Hodge-Tate type \mathbf{v} .*

Proof. It follows immediately from Lemma 3.11. \square

The goal of this subsection is to prove the following theorem:

Theorem 3.13. (cf. [24, Theorem 4.3.4]) *There exists a constant c_1 depending only on K', r , and d such that the following statement holds:*

Let A and A' be finite flat \mathcal{O}_E -algebras and let $\rho : \mathcal{G}_K \rightarrow \mathrm{GL}_d(A)$ and $\rho' : \mathcal{G}_K \rightarrow \mathrm{GL}_d(A')$ be Galois representations such that $\rho \in \mathrm{Rep}_A^{\mathrm{pst}, K', r}$ and $\rho' \in \mathrm{Rep}_{A'}^{\mathrm{pst}, K', r}$. Suppose that there exist an ideal $I \subset A$ such that A/I is killed by a power of p and an \mathcal{O}_E -linear surjection $\beta : A' \twoheadrightarrow A/I$ such that $A/I \otimes_A \rho \cong \beta' \circ \rho'$ as $A[\mathcal{G}_K]$ -modules. Here, $\beta' : \mathrm{GL}_d(A') \rightarrow \mathrm{GL}_d(A/I)$ is the natural map induced by β . Let V be the free $A[\frac{1}{p}]$ -module of rank d equipped with \mathcal{G}_K -action corresponding to $\rho \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, and similarly let V' corresponding to $\rho' \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. If $I \subset p^{c_1} A$ and V' has p -adic Hodge-Tate type \mathbf{v} , then V also has p -adic Hodge-Tate type \mathbf{v} .

When k is further assumed to be finite, Theorem 3.13 is proved in [24, Theorem 4.3.4]. Some arguments in the proof in [24] are based on reducing to the case when E contains the Galois closure of K' , and thus require k to be finite. We remove such a restriction by the following more general argument.

Since E is a finite extension of \mathbb{Q}_p , we have a ring isomorphism $E_K = E \otimes_{\mathbb{Q}_p} K \cong \prod_{j=1}^s H_j$ for some field H_j finite over K . Note that each H_j is an E_K -algebra via $E_K \cong \prod_{i=1}^s H_i \xrightarrow{q_j} H_j$ where q_j is the natural projection onto the j -th factor. For any E_K -module M , we write $M_j := M \otimes_{E_K} H_j$. Then $M \cong \bigoplus_{j=1}^s M_j$. For a filtered E_K -module D_K , we denote $(\mathrm{Fil}^i D_K)_j$ and $(\mathrm{gr}^i D_K)_j$ by $\mathrm{Fil}_j^i D_K$ and $\mathrm{gr}_j^i D_K$ respectively. Since any finite E_K -module is projective and thus flat, we have $\mathrm{gr}_j^i D_K \cong \mathrm{Fil}_j^i D_K / \mathrm{Fil}_j^{i+1} D_K$. We write $B_{H_j} := B \otimes_E H_j$.

Lemma 3.14. (cf. [24, Lemma 4.1.4]) *With notations as above, V_B has p -adic Hodge-Tate*

type \mathbf{v} if and only if $\mathrm{gr}_j^i D_{\mathrm{dR}}^K(V_B)$ is B_{H_j} -free and $\mathrm{rank}_{B_{H_j}} \mathrm{gr}_j^i D_{\mathrm{dR}}^K(V_B) = \dim_{H_j} \mathrm{gr}_j^i D_{E,K}$ for all $j = 1, \dots, s$ and $i \in \mathbb{Z}$.

Proof. This follows by the same argument as in the proof of [24, Lemma 4.1.4]. \square

$B_{\mathrm{red}} = N/\mathcal{N}(B)$ is a reduced Artinian E -algebra, so $B_{\mathrm{red}} \cong \prod_{l=1}^m E_l$ for some field E_l finite over E . We set $V_{E_l} = E_l \otimes_B V_B$.

Lemma 3.15. (cf. [24, Proposition 4.1.5]) *V_B has p -adic Hodge-Tate type \mathbf{v} if and only if V_{E_l} has p -adic Hodge-Tate type \mathbf{v} for each $l = 1, \dots, m$.*

Proof. This follows by the same argument as in the proof of [24, Proposition 4.1.5]. \square

The following lemma is needed when we consider an extension of the coefficient field E .

Lemma 3.16. *Let H be a field, and let C be a field (possibly infinite) over H . Let H' be a finite extension of H , and let R and T be finite extensions of H' . If M is a $C \otimes_H R$ -module such that $M \otimes_{H'} T$ is a finite free $C \otimes_H R \otimes_{H'} T$ -module, then M is finite free over $C \otimes_H R$.*

Proof. M is a finite projective module over $C \otimes_H R$, and there exists a surjection $f : M \otimes_{H'} T \twoheadrightarrow M$ of $C \otimes_H R$ -modules having a section. Let $\{e_1, \dots, e_n\}$ be a basis of $M \otimes_{H'} T$ over $C \otimes_H R \otimes_{H'} T$. Let $N := \bigoplus_{i=1}^n (C \otimes_H R) \cdot f(e_i)$. Then the natural map $N \rightarrow M$ of $C \otimes_H R$ -modules is an injection since $\{e_1, \dots, e_n\}$ is a basis of $M \otimes_{H'} T$ over $C \otimes_H R \otimes_{H'} T$. Furthermore, $\dim_C N = \dim_C M$, so it is bijective. \square

Let L be a finite extension of E , and write $B_L := L \otimes_E B$. Given \mathbf{v} as above, let $\mathbf{v}' = (D_L := L \otimes_E D, \{\mathrm{Fil}^i D_{L,K} = L \otimes_E \mathrm{Fil}^i D_{E,K}\}_{i=0, \dots, r})$.

Lemma 3.17. (cf. [24, Lemma 4.1.6]) *With notations as above, V_B has p -adic Hodge-Tate type \mathbf{v} if and only if $V_{B_L} := B_L \otimes_B V_B$ has p -adic Hodge-Tate type \mathbf{v}' .*

Proof. Given Lemma 3.16, it follows from the same argument as in the proof of [24, Lemma 4.1.6]. \square

Lemma 3.18. *Suppose we have an injection $B \hookrightarrow B'$ of finite E -algebras. If $V_{B'} = B' \otimes_B V_B$ has p -adic Hodge-Tate type \mathbf{v} , then also V_B has p -adic Hodge-Tate type \mathbf{v} .*

Proof. We have an induced injection of finite E -algebras $B_{\text{red}} \hookrightarrow B'_{\text{red}}$. By Lemma 3.15, we can reduce to the case when B and B' are fields. Then it follows from Lemma 3.14 and Lemma 3.16. \square

As we will apply the functor M_{st} to representations semi-stable over K' , we need to consider $D_{\text{dR}}^{K'}(V_B) := (B_{\text{dR}} \otimes_{\mathbb{Q}_p} V_B^\vee)^{\mathcal{G}_{K'}}$. Note that $D_{\text{dR}}^{K'}(V_B) = D_{\text{dR}}^K(V_B) \otimes_K K'$. Thus, by essentially the same argument as in the proof of Lemma 3.17, we see that V_B has p -adic Hodge-Tate type \mathbf{v} if and only if $\text{gr}^i D_{\text{dR}}^{K'}(V_B) \cong B \otimes_E \text{gr}^i D_{E,K'}$ as $B_{K'}$ -modules for all i . Here, $D_{E,K'} := D_E \otimes_E K' = D_{E,K} \otimes_K K'$ which has the induced filtration from $D_{E,K}$. Furthermore, we have a ring isomorphism $E_{K'} \cong \prod_{i=1}^n F_i$ for F_i finite extensions of K' , and the statement analogous to Lemma 3.14 holds for $D_{\text{dR}}^{K'}(V_B)$.

Let $\mathcal{O}_{E,K'} := \mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathcal{O}_{K'}$. The projection $q_i : E_{K'} \rightarrow F_i$ induces the map $q_i : \mathcal{O}_{E,K'} \rightarrow \mathcal{O}_{F_i}$. We then have the natural map $q : \mathcal{O}_{E,K'} \rightarrow \prod_{i=1}^n \mathcal{O}_{F_i}$. Denote by v_p the p -adic valuation normalized by $v_p(p) = 1$.

Lemma 3.19. *There exists a positive integer c' depending only on K_0 and $F(u)$ such that $p^{c'}(\prod_{i=1}^n \mathcal{O}_{F_i}) \subset q(\mathcal{O}_{E,K'})$.*

Proof. Let K_1 be the maximal unramified subextension over \mathbb{Q}_p contained in E . Then $K_1 = W(k_1)[\frac{1}{p}]$ for its residue field k_1 , and E/K_1 is totally ramified with a uniformizer ϖ_E and an Eisenstein polynomial $\tilde{F}(u)$ over K_1 . We have a ring isomorphism $k_1 \otimes_{\mathbb{F}_p} k \cong \prod_{j=1}^m l_j$ for some fields l_j finite over k . Let $G(u)$ be an irreducible polynomial satisfying

$K_1 \cong \mathbb{Q}_p[u]/G(u)$, and let $G(u) = \prod_{j=1}^m G_j(u)$ be the decomposition into irreducible factors over K_0 corresponding to the above decomposition of $k_1 \otimes_{\mathbb{F}_p} k$. Take a root θ_j of $G_j(u)$. Then, we have the isomorphisms

$$K_1 \otimes_{\mathbb{Q}_p} K_0 \cong \prod_{j=1}^m K_0(\theta_j) \cong \prod_{j=1}^m W(l_j) \left[\frac{1}{p} \right].$$

Note that the image of $W(k_1) \otimes_{\mathbb{Z}_p} W(k)$ under the above isomorphism followed by the projection onto the j -th factor is $W(l_j)$. Furthermore, $v_p(\prod_{j \neq s} G_j(\theta_s)) = 0$ for each $s = 1, \dots, m$, so $(0, \dots, 0, 1, 0, \dots, 0) \in \prod_{j=1}^m W(l_j) \left[\frac{1}{p} \right]$ whose s -th component is 1 lies in the image of $W(k_1) \otimes_{\mathbb{Z}_p} W(k)$ under the above isomorphism. Therefore, the natural injection $W(k_1) \otimes_{\mathbb{Z}_p} W(k) \hookrightarrow \prod_{j=1}^m W(l_j)$ is also surjective, i.e., it is an isomorphism. Since $\tilde{F}(u)$ is irreducible over $W(l_j) \left[\frac{1}{p} \right]$ for each j , we have $E \otimes_{\mathbb{Q}_p} K_0 \cong \prod_{j=1}^m L_j$ and $\mathcal{O}_E \otimes_{\mathbb{Z}_p} W(k) \cong \prod_{j=1}^m \mathcal{O}_{L_j}$ where $L_j := (W(l_j) \left[\frac{1}{p} \right]) (\varpi_E)$.

Now, for a field L finite over K_0 , let $F(u) = \prod_{s=1}^t F_s(u)$ be the decomposition of $F(u)$ into irreducible factors over L , and choose a root ϖ_s of $F_s(u)$. Let L' be the field over K_0 generated by the coefficients of $F_s(u)$ for all $s = 1, \dots, t$. Then L' is the minimal field over K_0 such that $F(u)$ has the same decomposition into irreducible factors as above. We have

$$L' \otimes_{K_0} K' \cong \prod_{s=1}^t L'[u]/F_s(u) \cong L'(\varpi_s),$$

and similarly for L . Let $q'_s : L' \otimes_{K_0} K' \rightarrow L'(\varpi_s)$ be the composition of the above isomorphism followed by the projection onto the s -th factor, and let

$A_s = q'_s(\mathcal{O}_{L'} \otimes_{\mathbb{Z}_p} W(k)[\pi])$. Under the natural map $\mathcal{O}_{L'} \otimes_{\mathbb{Z}_p} W(k)[\pi] \hookrightarrow \prod_{s=1}^t A_s$, we have $\prod_{h \neq s} F_h(\pi)$ mapping to $(0, \dots, 0, \prod_{h \neq s} F_h(\varpi_s), 0, \dots, 0)$ whose components are 0 except the s -th component. Write $v_p(\prod_{h \neq s} F_h(\varpi_s)) = \frac{a}{b}$ for some relatively prime positive inte-

gers a, b . Then $(\prod_{h \neq s} F_h(\varpi_s))^b = p^a x$ for some $x \in L'(\varpi_s)$ with $v_p(x) = 0$, and for some positive integer a_1 , we have $p^{a_1} x \in q'_s(\mathcal{O}_{L'} \otimes_{\mathbb{Z}_p} W(k)[\pi])$. Thus, $(0, \dots, 0, p^{a+a_1}, 0, \dots, 0)$, whose components are 0 except the s -th component, lies in the image of $\mathcal{O}_{L'} \otimes_{\mathbb{Z}_p} W(k)[\pi]$ under the above isomorphism. Since $\mathcal{O}_{L'} \subset \mathcal{O}_L$, the same holds for L .

Repeating this argument for all s and considering all possible decompositions of $F(u)$ into irreducible factors over some finite field over K_0 , we see that there exists a positive integer c' depending only on K_0 and $F(u)$ such that for any L finite over K_0 , if we write $L \otimes_{K_0} K \cong \prod_{s=1}^t L(\varpi_s)$ as above, then for each s , $(0, \dots, 0, p^{c'}, 0, \dots, 0)$ whose components are 0 except the s -th component lies in the image of $\mathcal{O}_L \otimes_{\mathbb{Z}_p} W(k)[\pi]$. Applying this for each L_j , we get the result. \square

Corollary 3.20. *Let M be a torsion free module over $\mathcal{O}_{E,K'}$. Then, for each $j = 1, \dots, n$, the torsion part of $M_j := M \otimes_{\mathcal{O}_{E,K'}, q_j} \mathcal{O}_{F_j}$ is killed by $p^{c'}$, where c' is the constant given in Lemma 3.19.*

Proof. Let $M' = \bigoplus_{j=1}^n M_j$. By Lemma 3.19, there exist morphisms of $\mathcal{O}_{E,K'}$ -modules $q_M : M \rightarrow M'$ and $s_M : M' \rightarrow M$ such that $q_M \circ s_M = p^{c'} \text{Id}|_{M'}$. Let x be a torsion element in M' . Then $s_M(x) = 0$, so $p^{c'} x = q_M(s_M(x)) = 0$. \square

Let C be a finite flat \mathcal{O}_E -algebra, and let $\Lambda \in \text{Rep}_C^{\text{pst}, K', r}$ such that Λ is a finite free C -module of rank d and $\Lambda[\frac{1}{p}]$ has Hodge-Tate type \mathbf{v} . Suppose there exist an ideal $J \subset C$ and a positive integer m such that $C/J \cong \mathcal{O}_E/p^m$. Suppose further that C is a local ring and there exists a prime ideal $\mathfrak{p} \subset C$ such that $C/\mathfrak{p} \cong \mathcal{O}_F$ for some finite extension F/\mathbb{Q}_p . Let $L_{K'} := M_{\text{st}}(\Lambda)_{K'}$.

Lemma 3.21. *$L_{K'}$ is finite free over $C_{K'} := C \otimes_{\mathbb{Z}_p} \mathcal{O}_{K'}$ of rank d .*

Proof. Let $\mathfrak{M} \in \text{Mod}_{\mathfrak{G}}^r(\varphi, \hat{\mathcal{G}})$ be the unique Kisin module such that $\hat{T}^\vee(\hat{\mathfrak{M}}) = \Lambda$ as given by

Theorem 3.6. Write $\mathfrak{S}_C := C \otimes_{\mathbb{Z}_p} \mathfrak{S}$. From the construction of the functor M_{st} in Section 3.2.2, it suffices to show that \mathfrak{M} is a finite free \mathfrak{S}_C -module of rank d .

The Kisin module corresponding to $\mathcal{O}_F \otimes_C \Lambda$ (via $C/\mathfrak{p} \cong \mathcal{O}_F$) is $\mathfrak{M}' := \mathcal{O}_F \otimes_C \mathfrak{M}$. Since \mathfrak{M}' is finite flat over \mathfrak{S} , $\mathfrak{M}'/u\mathfrak{M}'$ is p -torsion free by the Auslander-Buchsbaum Theorem. Thus, $\mathfrak{M}'/u\mathfrak{M}'$ is a projective $\mathcal{O}_F \otimes_{\mathbb{Z}_p} W(k)$ -module. Since $(\mathfrak{M}'/u\mathfrak{M}')[\frac{1}{p}]$ is isomorphic to its pullback by φ and φ permutes the maximal ideals of $\mathcal{O}_F \otimes_{\mathbb{Z}_p} W(k)$ transitively, we have that $\mathfrak{M}'/u\mathfrak{M}'$ is finite free over $\mathcal{O}_F \otimes_{\mathbb{Z}_p} W(k)$ of rank d . Thus, \mathfrak{M}' is finite free $\mathcal{O}_F \otimes_{\mathbb{Z}_p} \mathfrak{S}$ -module of rank d .

By Nakayama's lemma, we have a surjection

$$f : \bigoplus_{i=1}^d \mathfrak{S}_C \cdot e_i \twoheadrightarrow \mathfrak{M}$$

of \mathfrak{S}_C -modules. Λ is a finite free \mathbb{Z}_p -module of rank $[C : \mathbb{Z}_p]d$, so \mathfrak{M} is finite free over \mathfrak{S} of rank $[C : \mathbb{Z}_p]d$. Thus, f is an isomorphism. \square

For $s = 1, \dots, n$, we set $C[\frac{1}{p}]_s := (C[\frac{1}{p}] \otimes_{\mathbb{Q}_p} K') \otimes_{E_{K'}, q_s} F_s$, and define $d_s := \text{rank}_{C[\frac{1}{p}]_s}(\text{gr}_s^0(D_{\text{dR}}^{K'}(\Lambda[\frac{1}{p}])))$. Denote $\text{Fil}_s^i L_{K'} := \text{Fil}^i L_{K'} \otimes_{\mathcal{O}_{E, K'}, q_s} \mathcal{O}_{F_s}$, and similarly for the graded modules. By Lemma 3.21, $\text{Fil}_s^0 L_{K'}$ is free over $C_s := C_{K'} \otimes_{\mathcal{O}_{E, K'}, q_s} \mathcal{O}_{F_s}$ of rank d .

Lemma 3.22. (cf. [24, Lemma 4.2.7]) *Suppose that $d_s \neq 0$. Let l be a positive integer satisfying $m \geq ld + 1$. Then there exists $x \in \text{gr}_s^0 L_{K'}/J \text{gr}_s^0 L_{K'}$ such that $p^l x \neq 0$.*

Proof. This follows by essentially the same argument as in the proof of [24, Lemma 4.2.7], as follows. Denote M/JM by M/J for any C -module M . We have the following right exact sequence:

$$\text{Fil}_s^1 L_{K'} \rightarrow \text{Fil}_s^0 L_{K'} \rightarrow \text{gr}_s^0 L_{K'} \rightarrow 0.$$

Let $\tilde{\text{Fil}}_s^1 L_{K'}$ be the image of $\text{Fil}_s^1 L_{K'}$ in $\text{Fil}_s^0 L_{K'}$ under the first map in the above sequence.

We then obtain the following right exact sequence

$$\tilde{\text{Fil}}_s^1 L_{K'}/J \rightarrow \text{Fil}_s^0 L_{K'}/J \rightarrow \text{gr}_s^0 L_{K'}/J \rightarrow 0.$$

Denote $\bar{M} := \text{Fil}_s^0 L_{K'}/J$ and let $\bar{N} \subset \bar{M}$ be the submodule given by the image of $\tilde{\text{Fil}}_s^1 L_{K'}/J$.

Then $\bar{M}/\bar{N} := \text{gr}_s^0 L_{K'}/J$.

Suppose that p^l annihilates \bar{M}/\bar{N} . By Lemma 3.21, \bar{M} is a finite free \mathcal{O}_{F_s}/p^m -module of rank d . Let ϖ_s be a uniformizer of \mathcal{O}_{F_s} . Then there exists an \mathcal{O}_{F_s}/p^m -basis $\bar{e}_1, \dots, \bar{e}_d$ of \bar{M} such that

$$\bar{N} \cong \bigoplus_{i=1}^d \mathcal{O}_{F_s}/p^m \cdot (\varpi_s^{a_i} \bar{e}_i)$$

for some nonnegative integers a_i . We have $\varpi_s^{a_i} \mid p^l$ for all $i = 1, \dots, d$. Let e_1, \dots, e_d be a C_s -basis of $\text{Fil}_s^0 L_{K'}$ which lifts $\bar{e}_1, \dots, \bar{e}_d$. For $i = 1, \dots, d$, let $y_i \in \tilde{\text{Fil}}_s^1 L_{K'}$ which lifts $\varpi_s^{a_i} \bar{e}_i$. If X denotes the $d \times d$ -matrix such that $(y_1, \dots, y_d) = (e_1, \dots, e_d)X$, then $\det(X) = \varpi_s^a + b$ with $a = \sum_{i=1}^d a_i$ and $b \in J$. Since $m \geq ld + 1$, we have $\varpi_s^a \neq 0$ in C_s/J , and thus $\det(X) \neq 0$ in C_s . On the other hand, let $\bar{z}_1, \dots, \bar{z}_{d_s}$ be a $C[\frac{1}{p}]_s$ -basis of $\text{gr}_s^0(D_{\text{dR}}^{K'}(\Lambda[\frac{1}{p}]))$. We have $\det(X)(e_1, \dots, e_d) \subset \text{Fil}_s^1(D_{\text{dR}}^{K'}(\Lambda[\frac{1}{p}]))$, and therefore $\det(X)\bar{z}_i = 0$. This gives a contradiction.

□

Proof of Theorem 3.13. Given above results, the theorem follows from essentially the same argument as in the proof of [24, Theorem 4.3.4], except that we do not reduce to the case where E contains the Galois closure of K' . We recall the necessary arguments from [24] in the following.

We first reduce to the case where $A = \mathcal{O}_E$ and A' is local. For this, let $B := A \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

We have $B_{\text{red}} = B/\mathcal{N}(B) \cong \prod_j E_j$ with E_j finite over E . Let L be a finite Galois extension of E containing all Galois closures of E_j . Denote $\mathcal{O}_L \otimes_{\mathcal{O}_E} (*)$ by $(*)_{\mathcal{O}_L}$ for $(*)$ being A, A', ρ, ρ', I , and β . Note that $(A_{\mathcal{O}_L}[\frac{1}{p}])_{\text{red}} = L \otimes_E B_{\text{red}} = L \otimes_E \prod E_j \cong \prod_i L$ with E_j embedding into L differently. This induces the natural map $\psi_l : A_{\mathcal{O}_L} \rightarrow (A_{\mathcal{O}_L})[\frac{1}{p}] \twoheadrightarrow L$ to the l -th factor of $\prod_i L$. By Lemma 3.15 and 3.17, it suffices to show (assuming $I \subset p^{c_1} A$ for a suitable constant c_1) that $L \otimes_{\psi_l, A_{\mathcal{O}_L}} \rho$ has p -adic Hodge-Tate type \mathbf{v} . Let $A_l = \psi_l(A_{\mathcal{O}_L})$ and $I_l = \psi_l(I_{\mathcal{O}_L})$. $\psi_l : A_L \twoheadrightarrow A_l \subset L$ is a morphism of \mathcal{O}_L -algebras, so $A_l = \mathcal{O}_L$, and we have a natural projection $\gamma_l : A_{\mathcal{O}_L}/I_{\mathcal{O}_L} \twoheadrightarrow A_l/I_l$. Similarly, we can assume that $A'_{\mathcal{O}_L}$ admits a surjection onto \mathcal{O}_L . Thus, by replacing E by L and the others accordingly, we can assume that $A = \mathcal{O}_E$ and that A' admits a surjection onto \mathcal{O}_E . After localizing A' and β by a maximal ideal containing $\beta^{-1}(\varpi_E)$ where ϖ_E is a uniformizer of \mathcal{O}_E , we can further assume that A' is local.

Let T denote the torsion representation $A/I \otimes_A \rho \cong A'/I' \otimes_{A'} \rho' \in \text{Rep}_{\text{tor}, \mathcal{O}_E}^{\text{pst}, K', r}$ where $I' = \ker(\beta)$. We denote by j and j' the two lifts ρ and ρ' of T respectively. Write $L_{K'} := M_{\text{st}}(\rho)_{K'}$, $L'_{K'} := M_{\text{st}}(\rho')_{K'}$, $M_{K'} := M_{\text{st}, j}(T)_{K'}$, and $M'_{K'} := M_{\text{st}, j'}(T)_{K'}$. We have $\text{gr}_s^i M_{K'} \cong \text{gr}_s^i L_{K'}/I \text{gr}_s^i L_{K'}$ and $\text{gr}_s^i M'_{K'} \cong \text{gr}_s^i L'_{K'}/I' \text{gr}_s^i L'_{K'}$. By Corollary 3.9 and Proposition 3.10, there exists morphism of \mathcal{O}_{F_s} -modules $g_s^i : \text{gr}_s^i M_{K'} \rightarrow \text{gr}_s^i M'_{K'}$ and $h_s^i : \text{gr}_s^i M'_{K'} \rightarrow \text{gr}_s^i M_{K'}$ such that $g_s^i \circ h_s^i = p^{2c} \text{Id}|_{\text{gr}_s^i M'_{K'}}$ and $h_s^i \circ g_s^i = p^{2c} \text{Id}|_{\text{gr}_s^i M_{K'}}$.

Now, we set $\tilde{c} = \tilde{c}(K', r, d) := (2c + c')d + 1$ where c and c' are given as in Theorem 3.8 and Lemma 3.19 respectively. Assume $I \subset p^{\tilde{c}} A = p^{\tilde{c}} \mathcal{O}_E$. We claim that if $\text{gr}_s^0(D_{\text{dR}}^{K'}(V')) \neq 0$, then $\text{gr}_s^0(D_{\text{dR}}^{K'}(V)) \neq 0$. Suppose otherwise. By Corollary 3.20, $\text{gr}_s^0 M_{K'}$ is killed by $p^{c'}$. But by Lemma 3.22, there exists $x \in \text{gr}_s^0 M'_{K'}$ such that $p^{c'+2c} x \neq 0$. This gives a contradiction since $p^{c'+2c} x = g_s^0(p^{c'} h_s^0(x))$.

On the other hand, let $B' := A'[\frac{1}{p}]$, and denote $d_0 = \dim_{F_s} \text{gr}_s^0(D_{\text{dR}}^{K'}(V))$. We claim (assuming $I \subset p^{\tilde{c}} \mathcal{O}_E$) that $d_0 \leq \dim_{F_s} \text{gr}_s^0(D_{\text{dR}}^{K'}(V'))$. For this, note that as an \mathcal{O}_{F_s} -module,

$\mathrm{gr}_s^0 L_{K'} = N_{\mathrm{tor}} \oplus N$ where N_{tor} is the torsion submodule of $\mathrm{gr}_s^0 L_{K'}$ and N is a finite free \mathcal{O}_{F_s} -module of rank d_0 . By Corollary 3.20,

$$\mathrm{gr}_s^0 M_{K'} \cong N_{\mathrm{tor}} \oplus \bigoplus_{i=1}^{d_0} \mathcal{O}_{F_s} / I \mathcal{O}_{F_s}.$$

Let $\bar{N} := p^{c'} \bigoplus_{i=1}^{d_0} \mathcal{O}_{F_s} / I \mathcal{O}_{F_s}$. Then $p^{c'} \mathrm{gr}_s^0 M_{K'} = \bar{N}$, again by Corollary 3.20, and therefore $h_s^0(g_s^0(\bar{N})) \cong \bigoplus_{i=1}^{d_0} p^{2c+c'} \mathcal{O}_{F_s} / I \mathcal{O}_{F_s}$. Since $p^{c'} \mathrm{gr}_s^0 L'_{K'}$ surjects onto $h_s^0(p^{c'} \mathrm{gr}_s^0 M'_{K'})$ and $g_s^0(\bar{N}) \subset p^{c'} \mathrm{gr}_s^0 M'_{K'}$, we have by Corollary 3.20 that the \mathcal{O}_{F_s} -rank of $p^{c'} \mathrm{gr}_s^0 L'_{K'}$ is at least d_0 . Thus, the \mathcal{O}_{F_s} -rank of $\mathrm{gr}_s^0 L'_{K'}$ is at least d_0 , and $\dim_{F_s} \mathrm{gr}_s^0(D_{\mathrm{dR}}^{K'}(V')) \geq d_0$.

Hence, assuming $I \subset p^{\tilde{c}} \mathcal{O}_E$, we have $\mathrm{gr}_s^0(D_{\mathrm{dR}}^{K'}(V)) \neq 0$ if and only if $\mathrm{gr}_s^0(D_{\mathrm{dR}}^{K'}(V')) \neq 0$.

For the last step, we set $c_1 = \tilde{c}(K', dr, d)$ and assume $I \subset p^{c_1} \mathcal{O}_E$. It suffices to show that for each i ,

$$\dim_{F_s} \mathrm{gr}_s^i(D_{\mathrm{dR}}^{K'}(V)) = \mathrm{rank}_{B'_{F_s}} \mathrm{gr}_s^i(D_{\mathrm{dR}}^{K'}(V')).$$

Suppose that the above equation fails for some i , and let i_* be the smallest such number. Write $d_i = \dim_{F_s} \mathrm{gr}_s^i(D_{\mathrm{dR}}^{K'}(V))$ and $d'_i = \mathrm{rank}_{B'_{F_s}} \mathrm{gr}_s^i(D_{\mathrm{dR}}^{K'}(V'))$. Suppose first $d_{i_*} > d'_{i_*}$. We set $t_1 = \sum_{i \leq i_*} d_i$ and $t_2 = \sum_{i \leq i_*} i d_i$. Let $\tilde{i} = \max\{i \mid \sum_{j \leq i} d'_j \leq t_1\}$ and $t' = \sum_{i \leq \tilde{i}} d'_i$. Then $i_* \leq \tilde{i}$ and $t' \leq t_1$. Let

$$t'' = \left(\sum_{i \leq \tilde{i}} i d'_i \right) + (t_1 - t')(\tilde{i} + 1).$$

We have $t_2 < t''$. Moreover, t_2 (resp. t'') is the smallest i such that $\mathrm{gr}_s^i(D_{\mathrm{dR}}^{K'}(\bigwedge^{t_1} V))$ (resp. $\mathrm{gr}_s^i(D_{\mathrm{dR}}^{K'}(\bigwedge^{t_1} V'))$) is nontrivial. Let χ be a crystalline character such that $\mathrm{gr}_s^i(D_{\mathrm{dR}}^{K'}(\chi)) \neq 0$ only when $i = -t_2$. Then $\mathrm{gr}_s^0(D_{\mathrm{dR}}^{K'}(\chi \bigwedge^{t_1} V))$ is nontrivial. From the above result applied to $\chi \bigwedge^{t_1} V$ and $\chi \bigwedge^{t_1} V'$, we see that $\mathrm{gr}_s^0(D_{\mathrm{dR}}^{K'}(\chi \bigwedge^{t_1} V'))$ is also nontrivial, leading to a

contradiction.

By switching the roles of V and V' , it follows similarly that we cannot have $d_{i_*} < d'_{i_*}$.

This completes the proof. \square

3.3.2 Galois Type

We now study the Galois types of potentially semi-stable representations. As in the previous section, let E be a finite field over \mathbb{Q}_p , and let B be a finite E -algebra. Let V_B be a free B -module of rank d equipped with a potentially semi-stable \mathcal{G}_K -action. Let

$$D_{\text{pst}}(V_B) = \varinjlim_{K \subset K''} (B_{\text{st}} \otimes_{\mathbb{Q}_p} V_B^\vee)^{\mathcal{G}_{K''}}$$

where the limit goes over finite extensions of K contained in \bar{K} . Denote by K_0^{ur} the union of finite unramified extensions of K_0 contained in \bar{K} . We have $\dim_{K_0^{\text{ur}}} D_{\text{pst}}(V_B) = \dim_{\mathbb{Q}_p} V_B$.

Lemma 3.23. *Let B' be a finite B -algebra, and write $V_{B'} = B' \otimes_B V_B$. Then $V_{B'}$ is potentially semi-stable as a \mathcal{G}_K -representation, and $D_{\text{pst}}(V_{B'}) \cong B' \otimes_B D_{\text{pst}}(V_B)$. If V_B becomes semi-stable over $L \supset K$, then so does $V_{B'}$. Furthermore, $D_{\text{pst}}(V_B)$ is a finite free $B \otimes_{\mathbb{Q}_p} K_0^{\text{ur}}$ -module.*

Proof. It follows from essentially the same proof as for Lemma 3.11. \square

$D_{\text{pst}}(V_B)$ is equipped with a semilinear action of \mathcal{G}_K , and thus a linear action of the inertia group I_K . The Frobenius action commutes with I_K -action, so $\text{tr}(\sigma|D_{\text{pst}}(V_B)) \in B$ for all $\sigma \in I_K$.

Let D_E be an E -vector space of dimension d , and let $D_{E,K} = D_E \otimes_{\mathbb{Q}_p} K$ equipped with a filtration giving a p -adic Hodge-Tate type \mathbf{v} . Fix a representation

$$\tau : I_K \rightarrow \text{End}_E(D_E)$$

with an open kernel. Note that there exists an I_K -stable \mathcal{O}_E -lattice in D_E , so $\text{tr}(\tau(\sigma)) \in \mathcal{O}_E$ for all $\sigma \in I_K$. We say V_B has *Galois type* τ if the I_K -representation $D_{\text{pst}}(V_B)$ is equivalent to τ , i.e., $\text{tr}(\sigma|D_{\text{pst}}(V_B)) = \text{tr}(\tau(\sigma))$ for all $\sigma \in I_K$.

Let L/K be a finite Galois extension contained in \bar{K} such that $I_L \subset \ker(\tau)$. Here, I_L denotes the inertia subgroup of \mathcal{G}_L . $D_{\text{st}}^L(V_B) = (B_{\text{st}} \otimes_{\mathbb{Q}_p} V_B^\vee)^{\mathcal{G}_L}$ is an L_0 -vector space where L_0 is the maximal unramified subextension of K_0 contained in L . If V_B is semi-stable over L , then $D_{\text{pst}}(V_B) \cong K_0^{\text{ur}} \otimes_{L_0} D_{\text{st}}^L(V_B)$. Therefore, V_B has Galois type τ if and only if V_B becomes semi-stable over L and $\text{tr}(\sigma|D_{\text{st}}^L(V_B)) = \text{tr}(\tau(\sigma))$ for all $\sigma \in I_{L/K}$, where $I_{L/K}$ is the inertia subgroup of $\text{Gal}(L/K)$.

Lemma 3.24. *Let $\alpha : B \rightarrow B'$ be an E -algebra morphism between finite E -algebras. Suppose V is semi-stable over L . Then for all $\sigma \in I_{L/K}$, we have $\text{tr}(\sigma|D_{\text{st}}^L(V_{B'})) = \alpha(\text{tr}(\sigma|D_{\text{st}}^L(V_B)))$. In particular, if V_B has Galois type τ , then so does $V_{B'}$. If α is injective, then the converse is also true, i.e., V_B has Galois type τ if and only if $V_{B'}$ has Galois type τ .*

Proof. $D_{\text{pst}}(V_{B'}) \cong B' \otimes_B D_{\text{pst}}(V_B)$ by Lemma 3.23, so

$$\text{tr}(\sigma|D_{\text{st}}^L(V_{B'})) = \alpha(\text{tr}(\sigma|D_{\text{st}}^L(V_B)))$$

for all $\sigma \in I_{L/K}$. The remaining statements follow immediately. \square

Consider the case when B is local. If E' is its residue field, then E' is finite over E and B is naturally an E' -algebra. Note that the I_K -action on $D_{\text{pst}}(V_B)$ has an open kernel. Since the cohomology of a finite group with coefficients in $E' \otimes_{\mathbb{Q}_p} K_0^{\text{ur}}$ is trivial in all positive degrees, it follows from the deformation theory that the representation $D_{\text{pst}}(V_B)$ arises from a representation over $E' \otimes_{\mathbb{Q}_p} K_0^{\text{ur}}$. Thus, V_B has Galois type τ if and only if $V_{E'} = E' \otimes_B V_B$

has Galois type τ . For a general finite E -algebra B , we have isomorphisms $B \cong \prod_{i=1}^n B_{\mathfrak{m}_i}$ and $B_{\text{red}} \cong \prod_{i=1}^n E_i$, where $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ are the maximal ideals of B and $E_i = B_{\mathfrak{m}_i}/\mathfrak{m}_i B_{\mathfrak{m}_i}$. Let $V_{E_i} = E_i \otimes_B V_B$. We then have the following lemmas analogous to Lemma 3.15 and 3.17.

Lemma 3.25. *V_B has Galois type τ if and only if V_{E_i} has Galois type τ for each $i = 1, \dots, n$.*

Proof. It follows directly from Lemma 3.24. \square

Lemma 3.26. *Let E' be a finite extension of E , and let $B_{E'} = E' \otimes_E B$ and $V_{B_{E'}} = B_{E'} \otimes_B V_B$. Then V_B has Galois type τ if and only if $V_{B_{E'}}$ has Galois type τ .*

Proof. Since the natural map of E -algebras $B \rightarrow B_{E'}$ is injective, it follows from Lemma 3.24. \square

Together with Theorem 3.13, the following theorem is essential in proving the main result about potentially semi-stable deformation rings.

Theorem 3.27. *Let τ be a Galois type, and let L/K be a finite Galois extension in \bar{K} over which τ becomes trivial. Let A be a finite flat \mathcal{O}_E -algebra and $\rho : \mathcal{G}_K \rightarrow \text{GL}_d(A)$ be a Galois representation such that $\rho \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is semi-stable over L having Hodge-Tate weights in $[0, r]$.*

Suppose that for each positive integer n , there exist a finite flat \mathcal{O}_E -algebra A_n , a Galois representation $\rho_n : \mathcal{G}_K \rightarrow \text{GL}_d(A_n)$, and an \mathcal{O}_E -linear surjection $\beta_n : A_n \rightarrow A/p^n$ such that $A/p^n \otimes_A \rho \cong \beta'_n \circ \rho_n$ as $A[\mathcal{G}_K]$ -modules, and that $\rho_n \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is semi-stable over L having Hodge-Tate weights in $[0, r]$ and Galois type τ . Here, $\beta'_n : \text{GL}_d(A_n) \twoheadrightarrow \text{GL}_d(A/p^n)$ denotes the natural map induced by β_n .

Then $\rho \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ also has Galois type τ .

Proof. Let $B = A[\frac{1}{p}]$. We have $B_{\text{red}} \cong \prod_i E_i$ for some finite extensions E_i/E . Let H/E be a finite Galois extension containing the Galois closures of E_i for all i . We write $A_{\mathcal{O}_H} = \mathcal{O}_H \otimes_{\mathcal{O}_E} A$. Then $(A_{\mathcal{O}_H})_{\text{red}} \cong H \otimes_E B_{\text{red}} \cong H \otimes_E \prod_i E_i$. Since H contains the Galois closures of E_i for all i , $H \otimes_E E_i \cong \prod_j H$ with E_i embedding to H differently. This induces the natural map $\psi_l : A_{\mathcal{O}_H} \rightarrow A_{\mathcal{O}_H}[\frac{1}{p}] \rightarrow H$ to the l -th factor of $\prod_j H$. Let $A_l = \psi_l(A_{\mathcal{O}_H})$. Since $\psi_l : A_{\mathcal{O}_H} \rightarrow A_l \subset H$ is a morphism of \mathcal{O}_H -algebras, $A_l = \mathcal{O}_H$. By Lemma 3.25 and 3.26, it suffices to show that $H \otimes_{\psi_l, A_{\mathcal{O}_H}} (A_{\mathcal{O}_H} \otimes_A \rho)$ has Galois type τ . Therefore, we may and will replace ρ by the representation $\mathcal{O}_H \otimes_{\psi_l, A_{\mathcal{O}_H}} (A_{\mathcal{O}_H} \otimes_A \rho)$.

Denote by L_0 the maximal unramified extension of K_0 contained in L . Note that $I_{L/K} \cong I_{L/KL_0}$. Applying the results of Section 3.2 with (L_0, KL_0, L) in place of (K_0, K, K') , we get the associated lattice $M_{\text{st}}(\rho) \in L^r(\varphi, N, \text{Gal}(L/KL_0))$ in $D_{\text{st}}^L(\rho \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$. Since $M_{\text{st}}(\rho)$ is torsion free as $\mathcal{O}_H \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_0}$ -module, we see from the proofs of Lemma 3.19 and Corollary 3.20 that $M_{\text{st}}(\rho)$ is finite projective over $\mathcal{O}_H \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_0}$. Since $M_{\text{st}}(\rho)[\frac{1}{p}] = D_{\text{st}}^L(\rho \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ is a finite free $H \otimes_{\mathbb{Q}_p} L_0$ -module, $M_{\text{st}}(\rho)$ is finite free over $\mathcal{O}_H \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_0}$. Thus, for all $\sigma \in I_{L/K}$, $\text{tr}(\sigma | D_{\text{st}}^L(\rho \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)) = \text{tr}(\sigma | M_{\text{st}}(\rho)) \in \mathcal{O}_H$.

Now, fix a positive integer n , and let $B_n = A_n[\frac{1}{p}]$. $B_{n,\text{red}} \cong \prod_i F_i$ for some finite extensions F_i/E . Let H'/E be a finite Galois extension containing H and Galois closures of F_i for all i . Similarly as above, we have natural maps

$$\chi : \mathcal{O}_H \rightarrow \mathcal{O}_{H'} \otimes_{\mathcal{O}_E} \mathcal{O}_H \rightarrow \mathcal{O}_{H'}$$

such that if we set $\tilde{\rho} : \mathcal{G}_K \rightarrow \text{GL}_d(\mathcal{O}_{H'})$ to be the representation induced from ρ via χ , then it suffices to show that $\tilde{\rho} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ has Galois type τ . Note that for any $\sigma \in I_{L/K}$, $\text{tr}(\sigma | D_{\text{st}}^L(\tilde{\rho} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)) = \chi(\text{tr}(\sigma | D_{\text{st}}^L(\rho \otimes_{\mathbb{Z}_p} \mathbb{Q}_p))) \in \mathcal{O}_H$ since H/E is Galois. Likewise, we have

natural maps

$$A_n \rightarrow \mathcal{O}_{H'} \otimes_{\mathcal{O}_E} A_n \xrightarrow{\psi'} \mathcal{O}_{H'}.$$

ψ' has a section $s : \mathcal{O}_{H'} \hookrightarrow \mathcal{O}_{H'} \otimes_{\mathcal{O}_E} A_n$. Let $\gamma_n : (\mathcal{O}_{H'} \otimes_{\mathcal{O}_E} \beta_n) \circ s : \mathcal{O}_{H'} \twoheadrightarrow \mathcal{O}_{H'}/p^n$. We set $\tilde{\rho}_n : \mathcal{G}_K \rightarrow \mathrm{GL}_d(\mathcal{O}_{H'})$ be the Galois representation induced from ρ_n via the above map $A_n \rightarrow \mathcal{O}_{H'} \otimes_{\mathcal{O}_E} A_n \xrightarrow{\psi'} \mathcal{O}_{H'}$. Then $\tilde{\rho}_n \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ has Galois type τ , and $\mathcal{O}_{H'}/p^n \otimes_{\mathcal{O}_{H'}} \tilde{\rho} \cong \gamma'_n \circ \tilde{\rho}_n$ as $\mathcal{O}_{H'}[\mathcal{G}_K]$ -modules where $\gamma'_n : \mathrm{GL}_d(\mathcal{O}_{H'}) \rightarrow \mathrm{GL}_d(\mathcal{O}_{H'}/p^n)$ is the map induced by γ_n .

Denote by T the torsion \mathcal{G}_K -representation $\mathcal{O}_{H'}/p^n \otimes_{\mathcal{O}_{H'}} \tilde{\rho}$. T has two lifts j_1 and j_2 corresponding to $\tilde{\rho}$ and $\tilde{\rho}_n$ respectively, and we obtain

$M_{\mathrm{st},j_1}(T), M_{\mathrm{st},j_2}(T) \in M_{\mathrm{tor}}^{\mathrm{fl},r}(\varphi, N, \mathrm{Gal}(L/KL_0))$. Since $M_{\mathrm{st}}(\tilde{\rho})$ and $M_{\mathrm{st}}(\tilde{\rho}_n)$ are finite free $\mathcal{O}_{H'} \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_0}$ -modules of rank d , $M_{\mathrm{st},j_1}(T)$ and $M_{\mathrm{st},j_2}(T)$ are finite free over $\mathcal{O}_{H'}/p^n \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_0}$ of rank d , for which we fix a choice of bases. Let $\sigma \in I_{L/K}$, and for $i = 1, 2$, let C_i be the $d \times d$ matrix with coefficients in $\mathcal{O}_{H'}/p^n \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_0}$ which represents the σ -action on $M_{\mathrm{st},j_i}(T)$ with respect to the chosen bases. By Corollary 3.9 and Proposition 3.10, there exist $I_{L/K}$ -equivariant $\mathcal{O}_{H'} \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_0}$ -module morphisms $g_1 : M_{\mathrm{st},j_1}(T) \rightarrow M_{\mathrm{st},j_2}(T)$ and $g_2 : M_{\mathrm{st},j_2}(T) \rightarrow M_{\mathrm{st},j_1}(T)$ such that $g_1 \circ g_2 = p^{c''} \mathrm{Id}|_{M_{\mathrm{st},j_2}(T)}$ and $g_2 \circ g_1 = p^{c''} \mathrm{Id}|_{M_{\mathrm{st},j_1}(T)}$ where c'' is a constant depending only on the Eisenstein polynomial for L/L_0 and r . For $i = 1, 2$, let D_i be the $d \times d$ matrix with coefficients in $\mathcal{O}_{H'}/p^n \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_0}$ representing g_i . Then $D_1 D_2 = D_2 D_1 = p^{c''} \mathrm{Id}$ and $C_2 D_1 = D_1 C_1$. Thus,

$$\mathrm{tr}(C_2 D_1 D_2) = \mathrm{tr}(D_1 C_1 D_2) = \mathrm{tr}(D_2 D_1 C_1),$$

i.e., $p^{c''} \mathrm{tr}(C_1) = p^{c''} \mathrm{tr}(C_2)$ in $\mathcal{O}_{H'}/p^n$. Since $\mathrm{tr}(\sigma|M_{\mathrm{st}}(\tilde{\rho}_n)) = \mathrm{tr}(\tau(\sigma)) \in \mathcal{O}_E$ and $\mathrm{tr}(\sigma|M_{\mathrm{st}}(\tilde{\rho})) \in \mathcal{O}_H$, we have

$$\mathrm{tr}(\sigma|M_{\mathrm{st}}(\tilde{\rho})) - \mathrm{tr}(\tau(\sigma)) \in p^{\lfloor n-c'' \rfloor} \mathcal{O}_H.$$

Since this holds for all positive integers n , we have $\mathrm{tr}(\sigma|M_{\mathrm{st}}(\tilde{\rho})) = \mathrm{tr}(\tau(\sigma))$. \square

3.4 Construction of Galois Deformation Ring

We now construct the quotient of a given Galois deformation ring which corresponds to potentially semi-stable representations of a given p -adic Hodge-Tate type and Galois type. As before, let E/\mathbb{Q}_p be a finite extension with residue field \mathbb{F} . Let \mathcal{C} be the category of complete Noetherian local \mathcal{O}_E -algebras with residue field \mathbb{F} whose morphisms are local homomorphisms of \mathcal{O}_E -algebras that are identity on the residue field. Let \mathcal{C}^0 be the full subcategory of \mathcal{C} consisting of Artinian rings. We fix $(A_0, \mathfrak{m}_{A_0}) \in \mathcal{C}$ where \mathfrak{m}_{A_0} denotes the maximal ideal of A_0 , and let V_0 be a free A_0 -module of rank d equipped with A_0 -linear \mathcal{G}_K -action.

We fix a p -adic Hodge-Tate type \mathbf{v} and Galois type τ , and let L/K be a finite Galois extension over which τ becomes trivial. For $A \in \mathcal{C}^0$ and a \mathcal{G}_K -representation V_A induced from V_0 by a morphism $A_0 \rightarrow A$ in \mathcal{C} , we say V_A is *potentially semi-stable of type (\mathbf{v}, τ)* if there exist a finite flat \mathcal{O}_E -algebra $B \in \mathcal{C}$, a surjection $f : B \rightarrow A$ of \mathcal{O}_E -algebras in \mathcal{C} , and a free B -module V_B of rank d equipped with a B -linear \mathcal{G}_K -action such that $V_B \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is potentially semi-stable having p -adic Hodge-Tate type \mathbf{v} and Galois type τ , and $A \otimes_{f,B} V_B \cong V_A$.

Consider an assignment $D^{\mathbf{v}, \tau} : \mathcal{C}^0 \rightarrow \mathbf{Sets}$ associating to $A \in \mathcal{C}^0$ the set of representations induced from V_0 by morphisms $A_0 \rightarrow A$ in \mathcal{C} , which are potentially semi-stable of type (\mathbf{v}, τ) .

Lemma 3.28. *$D^{\mathbf{v}, \tau}$ is a functor.*

Proof. Let $R, S \in \mathcal{C}^0$ and let $\chi : R \rightarrow S$ be a morphism in \mathcal{C}^0 . It suffices to show that if $\rho \in D^{\mathbf{v}, \tau}(R)$, then $\chi \circ \rho \in D^{\mathbf{v}, \tau}(S)$. There exists a surjective ring homomorphism

$\chi' : R[x_1, \dots, x_n] \twoheadrightarrow S$ which extends χ such that $\chi'(x_i) \in \mathfrak{m}_S$ for each i . Let $I_{m,R} \subset R[x_1, \dots, x_n]$ denote the ideal generated by the m -th degree homogeneous polynomials with coefficients in R . Since S is Artinian, $\chi'(I_m) = 0$ for a sufficiently large m , and χ' induces a surjection $R[x_1, \dots, x_n]/I_m \twoheadrightarrow S$ for such m . Since ρ is potentially semi-stable of type (\mathbf{v}, τ) , there exist a finite flat \mathcal{O}_E -algebra $B \in \mathcal{C}$, a finite free B -module V_B , and a surjective morphism of \mathcal{O}_E -algebras $f : B \rightarrow R$ satisfying the conditions as above. f induces surjective homomorphisms of \mathcal{O}_E -algebras

$$f' : B' := B[x_1, \dots, x_n]/I_{m,B} \twoheadrightarrow R[x_1, \dots, x_n]/I_{m,R} \twoheadrightarrow S.$$

Note that $B' \in \mathcal{C}$, and all the above morphisms are in \mathcal{C} . Let $V_{B'} = B' \otimes_B V_B$. $V_{B'} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is semi-stable over L , and has p -adic Hodge-Tate type \mathbf{v} by Lemma 3.12. Moreover, by Lemma 3.24, it has Galois type τ . \square

We use Schlessinger's criteria in [33] to show that $D^{v,\tau}$ is pro-representable. Let $\mathbb{F}[T]/T^2 = \mathbb{F}[\epsilon]$ with ϵ being the image of T . A morphism $R \rightarrow S$ in \mathcal{C}^0 is called *small* if it is surjective such that the kernel is a principal ideal of R killed by \mathfrak{m}_R . Note that the projection $\mathbb{F}[\epsilon] \rightarrow \mathbb{F}$ is small.

For a functor $D' : \mathcal{C}^0 \rightarrow \mathbf{Sets}$ satisfying $|D'(\mathbb{F})| = 1$ and $R_0, R_1, R_2 \in \mathcal{C}^0$ with morphisms $f : R_1 \rightarrow R_0$ and $g : R_2 \rightarrow R_0$ in \mathcal{C}^0 , we have a natural map

$$(*) \quad D'(R_1 \times_{R_0} R_2) \rightarrow D'(R_1) \times_{D'(R_0)} D'(R_2),$$

where $R_1 \times_{R_0} R_2 = \{(a, b) \in R_1 \times R_2 \mid f(a) = g(b)\}$ and $D'(R_1) \times_{D'(R_0)} D'(R_2) = \{(a, b) \in D'(R_1) \times D'(R_2) \mid D'(f)(a) = D'(g)(b)\}$. Consider the following four conditions on D' :

H1. If $R_2 \rightarrow R_0$ is small, then $(*)$ is surjective.

H2. If $R_0 = \mathbb{F}$, $R_2 = \mathbb{F}[\epsilon]$, then for the natural projection $R_2 \rightarrow R_0$, $(*)$ is bijective.

H3. $D'(\mathbb{F}[\epsilon])$ is finite dimensional over \mathbb{F} .

H4. If $R_1 = R_2$ and $R_i \rightarrow R_0$ ($i = 1, 2$) are the same small map, then $(*)$ is bijective.

The following theorem is proved in [33] and [31].

Theorem 3.29. (cf. [33, Theorem 2.11], [31, Theorem 1.1])

1. **H1, H2, H3, H4** hold if and only if D' is pro-representable, i.e., representable by an object in \mathcal{C} .
2. Suppose D' is pro-representable and let D'' be a subfunctor of D' . Then D'' is pro-representable if and only if **H1** holds for D'' .

Proposition 3.30. The functor $D^{v,\tau}$ is pro-representable by a quotient $A_0^{v,\tau}$ of A_0 .

Proof. By Theorem 3.29 and [31, Proposition 1.2], it suffices to show that the map $(*)$ for $D^{v,\tau}$ is surjective when $R_2 \rightarrow R_0$ is small. Let $\tilde{\rho} \in D^{v,\tau}(R_1) \times_{D^{v,\tau}(R_0)} D^{v,\tau}(R_2)$ and write $R_3 := R_1 \times_{R_0} R_2$. There exists a representation $\rho : \mathcal{G}_K \rightarrow \mathrm{GL}_d(R_3)$ induced from V_0 by a morphism $A_0 \rightarrow R_3$ in \mathcal{C} which maps to $\tilde{\rho}$ under $(*)$. For $i = 1, 2$, let $\tilde{\rho}_i \in D^{v,\tau}(R_i)$ such that $(\tilde{\rho}_1, \tilde{\rho}_2) = \tilde{\rho} \in D^{v,\tau}(R_1) \times_{D^{v,\tau}(R_0)} D^{v,\tau}(R_2)$. The natural injection $R_3 \hookrightarrow R_1 \times R_2$ induces an injection of Galois representations $\tilde{\rho} \hookrightarrow \tilde{\rho}_1 \times \tilde{\rho}_2$. Since $\tilde{\rho}_i \in D^{v,\tau}(R_i)$, there exist finite flat \mathcal{O}_E -algebra $B_i \in \mathcal{C}$ lifting R_i and finite free B_i -module V_{B_i} lifting $\tilde{\rho}_i$ such that $V_{B_i} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is semi-stable over L having p -adic Hodge-Tate type v and Galois type τ . For the natural projection $q : B_1 \times B_2 \rightarrow R_1 \times R_2$, write $B := \{a \in B_1 \times B_2 \mid q(a) \in R_3\}$. Then B is a finite flat \mathcal{O}_E -algebra in \mathcal{C} , and the induced map $q : B \rightarrow R_3$ is a surjective \mathcal{O}_E -module morphism in \mathcal{C} . Furthermore, the representation $\mathcal{G}_K \rightarrow \mathrm{GL}_d(B_1 \times B_2)$ induced by $V_{B_1} \oplus V_{B_2}$ factors through $\mathrm{GL}_d(B)$, giving a representation V_B such that $R_3 \otimes_{q,B} V_B \cong \rho$.

By the main theorem for semi-stable representations in [25], there exists a quotient $B^{\text{st},L}$ of B associated to the locus of semi-stable representations over L induced from V_B . Since $(V_{B_1} \oplus V_{B_2}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is semi-stable over L and B injects into $B_1 \times B_2$, we have $B^{\text{st},L} = B$. Thus, $V_B \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is semi-stable over L . Furthermore, since $(V_{B_1} \oplus V_{B_2}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ has p -adic Hodge-Tate type \mathbf{v} and Galois type τ , $V_B \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ also has p -adic Hodge-Tate type \mathbf{v} and Galois type τ by Lemma 3.18 and 3.24. \square

We conclude this section by proving Theorem 1.3.

Proof of Theorem 1.3. Let $A := \rho(A_0) \subset B$. A is a finite flat \mathcal{O}_E -algebra in \mathcal{C} . Suppose first that $V_B := B \otimes_{\rho, A_0} V_0$ is potentially semi-stable having p -adic Hodge-Tate type \mathbf{v} and Galois type τ . Since A injects into B , we see for the induced representation $V_A := A \otimes_{\rho, A_0} V_0$ that $V_A \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is semi-stable over L . Furthermore, by Lemma 3.18 and 3.24, $V_A \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ has p -adic Hodge-Tate type \mathbf{v} and Galois type τ . So for every positive integer n , $A/p^n \otimes_A V_A \in D^{\mathbf{v},\tau}(A/p^n)$, and ρ factors through $A_0^{\mathbf{v},\tau}[\frac{1}{p}]$.

Conversely, suppose ρ factors through $A_0^{\mathbf{v},\tau}[\frac{1}{p}]$. For every positive integer n , $A/p^n \otimes_A V_A \in D^{\mathbf{v},\tau}(A/p^n)$. By the main theorem for semi-stable representations in [25], V_B is semi-stable over L . And by Theorem 3.13 and 3.27, V_B has p -adic Hodge-Tate type \mathbf{v} and Galois type τ . \square

4 Barsotti-Tate Representations in the Relative Case

4.1 Introduction

We keep the notations as in the previous sections (**except** that we will now define $\mathfrak{S} := R_0[[u]]$). Let R_0 be a ring which is unramified-good as in Definition 1.7. Let $R = R_0 \otimes_{W(k)} \mathcal{O}_K$ be the base ring for our relative setting. In the case $R = \mathcal{O}_K$, Raynaud showed the following theorem.

Theorem 4.1. (Raynaud [32, Proposition 2.3.1]) *Let $G = (G_n)_{n \geq 1}$ be a p -divisible group over K . Suppose that for each n , G_n extends to a finite locally free group scheme over \mathcal{O}_K . Then G extends to a p -divisible group over \mathcal{O}_K , and such an extension is unique up to isomorphism.*

In order to study Question 1.1 about the locus of Barsotti-Tate representations in the relative case, we formulate a generalized question for a higher dimensional base ring R .

Question 4.2. Let $G = (G_n)_{n \geq 1}$ be a p -divisible group over $R[\frac{1}{p}]$. Suppose that for each n , G_n extends to a finite locally free group scheme over R . Then does G extend to a p -divisible group over R ?

We first study the 2-dimensional cases when $R = \mathcal{O}_K[[t]]$ and when $R = \mathcal{O}_K\langle t^{\pm 1} \rangle$ which is the p -adic completion of the ring $\mathcal{O}_K[t^{\pm 1}]$. The proof of Theorem 4.1 given in [32] relies on the construction of scheme theoretic closure, which turns out to be a finite locally free group scheme when $R = \mathcal{O}_K$. However, even for the simplest higher dimensional case $R = \mathcal{O}_K[[t]]$, the analogue of scheme theoretic closure does not yield finite locally free group schemes in general. In fact, the answer to Question 4.2 is negative. We show that the failure of the higher dimensional version of Raynaud's theorem is closely related to the non-purity

result for a high ramification shown by Vasiu and Zink in [37]. Using the counter-example to the purity given in [37], we prove the following.

Theorem 4.3. *When $e \geq p$ and $R = \mathcal{O}_K[[t]]$, a p -divisible group G given as in Question 4.2 does not extend to R in general.*

Instead, we prove that the following weaker extendability holds.

Theorem 4.4. *Let $R = \mathcal{O}_K[[t]]$ and let G be as in Question 4.2. Then G extends to a p -divisible group over $\mathrm{Spec} R \setminus \{\mathfrak{m}\}$, where \mathfrak{m} is the maximal ideal of $\mathcal{O}_K[[t]]$. Furthermore, such an extension is unique up to isomorphism.*

When the ramification index is low, we have the positive result as stated in the following theorem:

Theorem 4.5. *In the situation of Question 4.2 with $R = \mathcal{O}_K[[t]]$ or $R = \mathcal{O}_K\langle t^{\pm 1} \rangle$, suppose further that the ramification index $e \leq p - 1$. Then G extends to a p -divisible group over R uniquely up to isomorphism.*

For higher dimensional cases with a low ramification, we prove the following theorem.

Theorem 4.6. *In the situation of Question 4.2 with $R = \mathcal{O}_K[[t_1, \dots, t_d]]$ or $R = \mathcal{O}_K\langle t_1^{\pm 1}, \dots, t_d^{\pm 1} \rangle$, suppose $e < p - 1$. Then G extends to a p -divisible group over R uniquely up to isomorphism.*

We will further see that Theorem 4.6 can be generalized to Theorem 1.8(2).

The main method for proving these results is the study of (torsion) Kisin modules of height 1. We make use of the generalizations of Breuil-Kisin classification of p -divisible groups and finite locally free group schemes in the relative case, which is established by Kim in [17]. Then, by considering certain “saturated” torsion modules over $\mathfrak{S} := R_0[[u]]$,

we generalize the weak full faithfulness of the functor $T_{\mathfrak{S}}$ mapping torsion Kisin modules to Galois representations, which is studied in [25] when $R = \mathcal{O}_K$.

When $R = \mathcal{O}_K \langle t_1^{\pm 1}, \dots, t_d^{\pm 1} \rangle$, it is necessary to take into account the topologically quasi-nilpotent integrable connections for torsion Kisin modules associated with finite locally free group schemes. To get a compatible system of such connections, we apply a finiteness result on the moduli of connections shown by Vasiu in [36].

The above results have an application in the study of Barsotti-Tate representations in the relative case. We prove Theorem 1.9 and Corollary 1.10, which in particular state that (assuming some conditions on the base scheme) being Barsotti-Tate is a Zariski-closed condition when the ramification is low.

4.2 Relative Breuil-Kisin Classification

4.2.1 Classification of p -divisible Groups and Finite Locally Free Group Schemes

We first explain the classification of p -divisible groups and finite locally free group schemes over R via certain Kisin modules, which is proved in [18] when $R = \mathcal{O}_K$ and generalized in [17] to the relative case.

Let R_0 be an unramified-good ring, and let $R = R_0 \otimes_{W(k)} \mathcal{O}_K$ as above so that R is good as in Definition 1.7.

Lemma 4.7. *R_0 and R_0/pR_0 are both regular.*

Proof. Let k' and J be as in Definition 1.7 so that R_0 is formally smooth formally finite type over $W(k')$ with respect to the J -adic topology. We have $J \subset \text{rad}(R_0)$ since R_0 is J -adically complete. Furthermore, R_0 is Noetherian since J is finitely generated.

Note that R_0/pR_0 is a formally smooth formally finite type k' -algebra with respect to the J/p -adic topology. Let $\mathfrak{M} \subset R_0/pR_0$ be a maximal ideal. Then $(R_0/pR_0)_{\mathfrak{M}}$ is

formally smooth over k' with respect to $\bar{\mathfrak{M}}(R_0/pR_0)_{\bar{\mathfrak{M}}}$ -adic topology. By [28, Lemma 28.1], $(R_0/pR_0)_{\bar{\mathfrak{M}}}$ is regular. Thus, R_0/pR_0 is regular.

We have $p \in \text{rad}(R_0)$. By [28, Theorem 28.9], R_0 is flat over $W(k)$. Let $\mathfrak{M} \subset R_0$ be a maximal ideal, and write $\bar{\mathfrak{M}} := \mathfrak{M}/p \subset R_0/pR_0$. Note that $(R_0)_{\mathfrak{M}}/p = (R_0/pR_0)_{\bar{\mathfrak{M}}}$ is a regular local ring and hence an integral domain by [28, Theorem 14.3]. Thus, we have the following identity of Krull dimensions:

$$\dim((R_0)_{\mathfrak{M}}/p) + 1 = \dim((R_0)_{\mathfrak{M}}).$$

From this, we conclude that $(R_0)_{\mathfrak{M}}$ is regular, and thus R_0 is regular. \square

For an \mathbb{F}_p -algebra A , we say that A locally has a *finite p -basis* if locally A is finite free as a module over itself via Frobenius, having a basis given by monomials $\{x^I \mid I = (i_\alpha), 0 \leq i_\alpha < p\}$ (cf. [9, Definition 1.1.1]).

Lemma 4.8. *R_0/pR_0 locally has a finite p -basis.*

Proof. Note that R_0/pR_0 is regular by Lemma 4.7. Since k is perfect and R_0/pR_0 is formally finite type with respect to a finitely generated ideal, R_0/pR_0 is finite as a module over itself via Frobenius. Thus, by [22, Theorem 15.7] and [39, Lemma 7], R_0/pR_0 locally has a finite p -basis. \square

By Lemma 4.8, R satisfies the p -basis assumption and formally finite-type assumption defined in [17, Section 2.2]. In particular, we can apply the results in [17] for classifying p -divisible groups and finite locally free group schemes over R .

Typical examples are $R = \mathcal{O}_K[[t_1, \dots, t_d]]$ or $R = \mathcal{O}_K\langle t_1^{\pm 1}, \dots, t_d^{\pm 1} \rangle$. Denote by $\hat{\Omega}_{R_0} := \varprojlim_n \Omega_{(R_0/p^n)/\mathbb{Z}_p}$ the module of p -adically continuous Kahler differentials, and let $\mathfrak{S} = R_0[[u]]$.

By [17, Lemma 2.3.1], the natural Frobenius on $W(k)$ extends (not necessarily uniquely) to R_0 . We fix such a Frobenius morphism φ on R_0 . Let r be a positive integer.

Definition 4.9. A *Kisin module of height r* is a pair $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ where

- \mathfrak{M} is a finitely generated projective \mathfrak{S} -module, and
- $\varphi_{\mathfrak{M}} : \mathfrak{M} \rightarrow \mathfrak{M}$ is a φ -semilinear map such that $\text{coker}(1 \otimes \varphi_{\mathfrak{M}})$ is killed by $E(u)^r$.

A morphism between two Kisin modules $(\mathfrak{M}_1, \varphi_{\mathfrak{M}_1})$ and $(\mathfrak{M}_2, \varphi_{\mathfrak{M}_2})$ is a morphism as \mathfrak{S} -modules which is compatible with $\varphi_{\mathfrak{M}_i}$. Let $\text{Mod}_{\mathfrak{S}}^r(\varphi)$ denote the category of Kisin modules of height r .

Let pc_0 be the constant term of $E(u)$. If $(\mathfrak{M}, \varphi_{\mathfrak{M}}) \in \text{Mod}_{\mathfrak{S}}^r(\varphi)$, then $1 \otimes \varphi_{\mathfrak{M}} : \varphi^* \mathfrak{M} \rightarrow \mathfrak{M}$ is injective since \mathfrak{M} is finite projective over \mathfrak{S} and $\text{coker}(1 \otimes \varphi_{\mathfrak{M}})$ is killed by $E(u)^r$. Thus, there exists a unique injective \mathfrak{S} -linear morphism $\psi_{\mathfrak{M}} : \mathfrak{M} \rightarrow \varphi^* \mathfrak{M}$ such that $(1 \otimes \varphi_{\mathfrak{M}}) \circ \psi_{\mathfrak{M}} = c_0^{-r} E(u)^r \text{Id}_{\mathfrak{M}}$ and $\psi_{\mathfrak{M}} \circ (1 \otimes \varphi_{\mathfrak{M}}) = c_0^{-r} E(u)^r \text{Id}_{\varphi^* \mathfrak{M}}$.

Let $\text{Mod}_{\mathfrak{S}}^r(\varphi, \nabla)$ denote the category whose objects are triples $(\mathfrak{M}, \varphi_{\mathfrak{M}}, \nabla_{\mathcal{M}})$ such that $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ is a Kisin module of height r , $\mathcal{M} := R_0 \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$, and $\nabla_{\mathcal{M}} : \mathcal{M} \rightarrow \mathcal{M} \otimes_{R_0} \hat{\Omega}_{R_0}$ is a topologically quasi-nilpotent integrable connection which commutes with $\varphi_{\mathcal{M}} := \varphi_{R_0} \otimes \varphi_{\mathfrak{M}}$. The following theorem is proved in [17].

Theorem 4.10. (cf. [17, Corollary 6.3.1 and 10.3.1, Remark 6.1.6 and 6.3.5]) *There exists an exact anti-equivalence of categories*

$$\mathfrak{M}^* : \{p\text{-divisible groups over } R\} \rightarrow \text{Mod}_{\mathfrak{S}}^1(\varphi, \nabla).$$

Let R'_0 be another unramified-good ring equipped with a Frobenius. Then the formation of \mathfrak{M}^* commutes with any base change $R \rightarrow R' := R'_0 \otimes_{W(k)} \mathcal{O}_K$ induced by a φ -equivariant morphism $R_0 \rightarrow R'_0$.

When $R_0 = W(k)[[t_1, \dots, t_d]]$ with φ given by $\varphi(t_i) = t_i^p$, the natural functor $\text{Mod}_{\mathfrak{S}}^1(\varphi, \nabla) \rightarrow \text{Mod}_{\mathfrak{S}}^1(\varphi)$ forgetting the connection is an equivalence of categories.

The classification of finite locally free group schemes over R is obtained by considering torsion Kisin modules.

Definition 4.11. A *torsion Kisin module of height r* is a pair $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ where

- \mathfrak{M} is a finitely presented \mathfrak{S} -module killed by some power of p , and of \mathfrak{S} -projective dimension ≤ 1 , and
- $\varphi_{\mathfrak{M}} : \mathfrak{M} \rightarrow \mathfrak{M}$ is a φ -semilinear map such that $\text{coker}(1 \otimes \varphi_{\mathfrak{M}})$ is killed by $E(u)^r$.

Let $\text{Mod}_{\mathfrak{S}}^{\text{tor}, r}(\varphi)$ denote the category of torsion Kisin modules of height r .

Lemma 4.12. Let $(\mathfrak{M}, \varphi_{\mathfrak{M}}) \in \text{Mod}_{\mathfrak{S}}^{\text{tor}, r}(\varphi)$. Then $1 \otimes \varphi_{\mathfrak{M}} : \varphi^* \mathfrak{M} \rightarrow \mathfrak{M}$ is injective.

Proof. Let R'_0 be the p -adic completion of the perfect closure $\varinjlim_{\varphi} (R_0)_{(p)}$. By the universal property of Witt vectors, we have a φ -equivariant isomorphism $R'_0 \cong W(k')$ where $k' := \varinjlim_{\varphi} \text{Frac}(R_0/p)$ denotes the perfect closure of $\text{Frac}(R_0/p)$. Let $\mathfrak{S}' := R'_0[[u]]$, and let $b_g : R_0 \rightarrow R'_0$ be the natural φ -equivariant map. Note that $\mathfrak{M}' := \mathfrak{M} \otimes_{\mathfrak{S}, b_g} \mathfrak{S}'$ equipped with $\varphi_{\mathfrak{M}'} := \varphi_{\mathfrak{M}} \otimes \varphi_{\mathfrak{S}'}$ is a Kisin module of height r over \mathfrak{S}' . Furthermore, we have the following commutative diagram.

$$\begin{array}{ccc} \varphi^* \mathfrak{M} & \xrightarrow{1 \otimes \varphi_{\mathfrak{M}}} & \mathfrak{M} \\ \downarrow & & \downarrow \\ \varphi^*(\mathfrak{M}') & \xrightarrow{1 \otimes \varphi_{\mathfrak{M}'}} & \mathfrak{M}' \end{array}$$

Since \mathfrak{M} has \mathfrak{S} -projective dimension 1 and is killed by some power of p , the right vertical map is injective. Since $\varphi : \mathfrak{S} \rightarrow \mathfrak{S}$ is flat by [6, Lemma 7.1.8], the left vertical map is also injective. Furthermore, by [25, Proposition 2.3.2], the bottom map is injective since $R'_0 \cong W(k')$. This implies that the top map is injective. \square

By Lemma 4.12, there exists a unique injective \mathfrak{S} -linear morphism $\psi_{\mathfrak{M}} : \mathfrak{M} \rightarrow \varphi^* \mathfrak{M}$ such that $(1 \otimes \varphi_{\mathfrak{M}}) \circ \psi_{\mathfrak{M}} = c_0^{-r} E(u)^r \text{Id}_{\mathfrak{M}}$ and $\psi_{\mathfrak{M}} \circ (1 \otimes \varphi_{\mathfrak{M}}) = c_0^{-r} E(u)^r \text{Id}_{\varphi^* \mathfrak{M}}$.

Let $\text{Mod}_{\mathfrak{S}}^{\text{tor}, r}(\varphi, \nabla)$ denote the category whose objects are triples $(\mathfrak{M}, \varphi_{\mathfrak{M}}, \nabla_{\mathcal{M}})$ such that $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ is a torsion Kisin module of height r , $\mathcal{M} := R_0 \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$, and $\nabla_{\mathcal{M}} : \mathcal{M} \rightarrow \mathcal{M} \otimes_{R_0} \hat{\Omega}_{R_0}$ is a topologically quasi-nilpotent integrable connection which commutes with $\varphi_{\mathcal{M}}$. The following proposition and theorems are shown in [17].

Proposition 4.13. (cf. [17, Proposition 9.3]) *There exists an exact fully faithful functor \mathfrak{M}^* from the category of p -power order finite locally free group schemes over R to $\text{Mod}_{\mathfrak{S}}^{\text{tor}, 1}(\varphi, \nabla)$ with the following properties:*

- *Let H be a finite locally free group scheme over R . If $H = \ker(d : G^0 \rightarrow G^1)$ for an isogeny d of p -divisible groups, then there exists a natural isomorphism $\mathfrak{M}^*(H) \cong \text{coker}(\mathfrak{M}^*(d))$ of torsion Kisin modules.*
- *Let R'_0 be another unramified-good ring equipped with a Frobenius. Then the formation of \mathfrak{M}^* commutes with any base change $R \rightarrow R' := R'_0 \otimes_{W(k)} \mathcal{O}_K$ induced by a φ -equivariant morphism $R_0 \rightarrow R'_0$.*

Theorem 4.14. (cf. [17, Theorem 9.4]) *When $R_0 = W(k)[[t_1, \dots, t_d]]$ with φ given by $\varphi(t_i) = t_i^p$, the functor \mathfrak{M}^* as in Proposition 4.13 composed with the natural functor $\text{Mod}_{\mathfrak{S}}^{\text{tor}, 1}(\varphi, \nabla) \rightarrow \text{Mod}_{\mathfrak{S}}^{\text{tor}, 1}(\varphi)$ forgetting the connection is an anti-equivalence of categories.*

We denote by $(\text{Mod FI})_{\mathfrak{S}}(\varphi, \nabla)$ the full subcategory of $\text{Mod}_{\mathfrak{S}}^{\text{tor}, 1}(\varphi, \nabla)$ consisting of \mathfrak{M} such that $\mathfrak{M} \cong \oplus \mathfrak{M}_i$ as a \mathfrak{S} -module where \mathfrak{M}_i are projective over \mathfrak{S}/p^{n_i} .

Theorem 4.15. (cf. [17, Theorem 9.4]) *\mathfrak{M}^* as in Proposition 4.13 induces an anti-*

equivalence of categories:

$$\mathfrak{M}^* : \left\{ \begin{array}{l} p\text{-power order finite locally} \\ \text{free group schemes } H \text{ over } R \\ \text{such that } H[p^n] \text{ is locally free for all } n \end{array} \right\} \rightarrow (\text{Mod FI})_{\mathfrak{S}}(\varphi, \nabla).$$

4.2.2 Étale φ -modules and Galois Representations

We summarize the results in [17, Section 7] about étale φ -modules in the relative setting, and associate Question 4.2 with a question about (torsion) Kisin modules of height 1. The underlying theory is based on Scholze's work on perfectoid spaces in [34].

We assume further that R is an integral domain. Let \bar{R} denote the union of normal R -subalgebras R' of a fixed separable closure of R such that $R'[\frac{1}{p}]$ is finite étale over $R[\frac{1}{p}]$. Then $\text{Spec} \bar{R}[\frac{1}{p}]$ is a pro-universal covering of $\text{Spec} R[\frac{1}{p}]$, and \bar{R} is an integral closure of R in $\bar{R}[\frac{1}{p}]$. Let $\hat{\bar{R}}$ be the p -adic completion of \bar{R} , and let $\mathcal{G}_R := \text{Gal}(\bar{R}[\frac{1}{p}]/R[\frac{1}{p}]) = \pi_1^{\text{ét}}(\text{Spec} R[\frac{1}{p}], \eta)$ where η is a geometric point we choose (the more precise notation in accordance with $\mathcal{G}_K = \text{Gal}(\bar{K}/K)$ would be $\mathcal{G}_{R[\frac{1}{p}]}$, but we use \mathcal{G}_R for brevity). We choose a compatible system of p^n -th roots ϖ_n of ϖ , and let L be the p -adic completion of $\bigcup_{n \geq 0} K_0(\varpi_n)$. Then L is a perfectoid field, and $(\hat{\bar{R}}[\frac{1}{p}], \hat{\bar{R}})$ is a perfectoid affinoid L -algebra.

Let $\underline{\varpi} := (\varpi_n) \in L^\flat$ where L^\flat denotes the tilt of L as defined in [34]. Let $(\bar{R}^\flat[\frac{1}{\underline{\varpi}}], \bar{R}^\flat)$ be the tilt of $(\hat{\bar{R}}[\frac{1}{p}], \hat{\bar{R}})$. Let $\mathbf{E}_{R_\infty}^+ := \mathfrak{S}/p$, and let $\tilde{\mathbf{E}}_{R_\infty}^+$ be the u -adic completion of $\varinjlim_{\varphi} \mathbf{E}_{R_\infty}^+$. Let $\mathbf{E}_{R_\infty} = \mathbf{E}_{R_\infty}^+[\frac{1}{u}]$ and $\tilde{\mathbf{E}}_{R_\infty} = \tilde{\mathbf{E}}_{R_\infty}^+[\frac{1}{u}]$. By [34, Proposition 5.9], $(\tilde{\mathbf{E}}_{R_\infty}, \tilde{\mathbf{E}}_{R_\infty}^+)$ is a perfectoid affinoid L^\flat -algebra, and we have the natural injection $(\tilde{\mathbf{E}}_{R_\infty}, \tilde{\mathbf{E}}_{R_\infty}^+) \hookrightarrow (\bar{R}^\flat[\frac{1}{\underline{\varpi}}], \bar{R}^\flat)$ given by $u \mapsto \underline{\varpi}$. Let $(\tilde{R}_\infty[\frac{1}{p}], \tilde{R}_\infty)$ be a perfectoid affinoid L -algebra whose tilt is $(\tilde{\mathbf{E}}_{R_\infty}, \tilde{\mathbf{E}}_{R_\infty}^+)$. Let $\mathcal{G}_{\tilde{R}_\infty}$ be the étale fundamental group of $\text{Spec} \tilde{R}_\infty[\frac{1}{p}]$ with the given geometric point η . Then, we have a continuous map $\mathcal{G}_{\tilde{R}_\infty} \rightarrow \mathcal{G}_R$, which is a closed embedding by [15, Proposi-

tion 5.4.54]. By the almost purity theorem in [34], $\bar{R}^b[\frac{1}{\varpi}]$ can be canonically identified with the ϖ -adic completion of the affine ring of a pro-universal covering of $\mathrm{Spec} \tilde{\mathbf{E}}_{R_\infty}$, and defining $\mathcal{G}_{\tilde{\mathbf{E}}_{R_\infty}}$ using this pro-universal covering, there is a canonical isomorphism $\mathcal{G}_{\tilde{\mathbf{E}}_{R_\infty}} \cong \mathcal{G}_{\bar{R}_\infty}$.

Now, let $\mathcal{O}_\mathcal{E}$ be the p -adic completion of $\mathfrak{S}[\frac{1}{u}]$, and let $\mathcal{E} = \mathcal{O}_\mathcal{E}[\frac{1}{p}]$. Note that φ on \mathfrak{S} extends naturally to \mathcal{E} .

Definition 4.16. An *étale* $(\varphi, \mathcal{O}_\mathcal{E})$ -module is a pair (M, φ_M) where M is a finitely generated $\mathcal{O}_\mathcal{E}$ -module and $\varphi_M : M \rightarrow M$ is a φ -semilinear endomorphism such that the linearization $1 \otimes \varphi_M : \varphi^* M \rightarrow M$ is an isomorphism. We say that an étale $(\varphi, \mathcal{O}_\mathcal{E})$ -module is projective (resp. torsion) if the underlying $\mathcal{O}_\mathcal{E}$ -module M is projective (resp. p -power torsion).

Let $\mathrm{Mod}_{\mathcal{O}_\mathcal{E}}^{\mathrm{ét}}(\varphi)$ denote the category of étale $(\varphi, \mathcal{O}_\mathcal{E})$ -modules. Let $\mathrm{Mod}_{\mathcal{O}_\mathcal{E}}^{\mathrm{ét}, \mathrm{pr}}(\varphi)$ and $\mathrm{Mod}_{\mathcal{O}_\mathcal{E}}^{\mathrm{ét}, \mathrm{tor}}(\varphi)$ respectively denote the full subcategories of projective and torsion objects.

Note that there exists a natural notion of a subquotient, direct sum, and tensor product for étale $(\varphi, \mathcal{O}_\mathcal{E})$ -modules. Duality is defined for projective and torsion objects. For $(\mathfrak{M}, \varphi_\mathfrak{M})$ a Kisin module (resp. torsion Kisin module) of height r , $(\mathfrak{M} \otimes_\mathfrak{S} \mathcal{O}_\mathcal{E}, \varphi_\mathfrak{M} \otimes \varphi_{\mathcal{O}_\mathcal{E}})$ is a projective (resp. torsion) étale $(\varphi, \mathcal{O}_\mathcal{E})$ -module, since $1 \otimes \varphi_\mathfrak{M}$ is injective and its cokernel is killed by $E(u)^r$.

We consider $W(\bar{R}^b[\frac{1}{\varpi}])$ as an $\mathcal{O}_\mathcal{E}$ -algebra via $u \mapsto [\varpi]$, and let $\mathcal{O}_\mathcal{E}^{\mathrm{ur}}$ be the integral closure of $\mathcal{O}_\mathcal{E}$ in $W(\bar{R}^b[\frac{1}{\varpi}])$. Let $\hat{\mathcal{O}}_\mathcal{E}^{\mathrm{ur}}$ be its p -adic closure, and let

$$\mathfrak{S}^{\mathrm{ur}} := \hat{\mathcal{O}}_\mathcal{E}^{\mathrm{ur}} \cap W(\bar{R}^b) \subset W(\bar{R}^b[\frac{1}{\varpi}]).$$

Since $\mathcal{O}_\mathcal{E}$ is normal, we have $\mathrm{Aut}_{\mathcal{O}_\mathcal{E}}(\mathcal{O}_\mathcal{E}^{\mathrm{ur}}) \cong \mathcal{G}_{\mathbf{E}_{R_\infty}}$, and by [15, Proposition 5.4.54] and the almost purity theorem, we have $\mathcal{G}_{\mathbf{E}_{R_\infty}} \cong \mathcal{G}_{\tilde{\mathbf{E}}_{R_\infty}} \cong \mathcal{G}_{\bar{R}_\infty}$. This induces $\mathcal{G}_{\bar{R}_\infty}$ -action on $\hat{\mathcal{O}}_\mathcal{E}^{\mathrm{ur}}$ and on $\mathfrak{S}^{\mathrm{ur}}$.

Remark 4.17. We use a different definition of \mathfrak{S}^{ur} from the one used in [17, Section 7]. The definition we give here is the one used in [25] and [18] for the case $R = \mathcal{O}_K$. The reason is that we need Lemma 4.26, for which the proof relies on Lemma 4.24.

Lemma 4.18. *We have $\mathcal{O}_{\mathcal{E}} = (\hat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}})^{\mathcal{G}_{\tilde{R}_{\infty}}}$, and $\mathfrak{S} = (\mathfrak{S}^{\text{ur}})^{\mathcal{G}_{\tilde{R}_{\infty}}}$. The same holds modulo p^n .*

Proof. By [17, Lemma 7.2.6], we have $\mathcal{O}_{\mathcal{E}} = (\hat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}})^{\mathcal{G}_{\tilde{R}_{\infty}}}$, and the same statement holds modulo p^n . Hence, it suffices to show that $(\mathfrak{S}^{\text{ur}})^{\mathcal{G}_{\tilde{R}_{\infty}}} \subset \mathfrak{S}$.

Note that the perfectoid algebra $\bar{R}^b[\frac{1}{\varpi}]$ is equipped with the rank-1 ϖ -adic valuation $|\cdot|_{\varpi}$ satisfying $|\varpi|_{\varpi} = |\varpi|$ (cf. [34, Lemma 3.4 and Section 5]). For $x \in \mathcal{O}_{\mathcal{E}}/p$, we have $x \in \mathfrak{S}/p$ if and only if $|x|_{\varpi} \leq 1$. Thus, $\mathcal{O}_{\mathcal{E}}/p \cap \bar{R}^b = \mathfrak{S}/p$.

Hence, $\mathcal{O}_{\mathcal{E}} \cap W(\bar{R}^b) = \mathfrak{S}$, which implies that $(\mathfrak{S}^{\text{ur}})^{\mathcal{G}_{\tilde{R}_{\infty}}} \subset \mathfrak{S}$. \square

Lemma 4.19. *There exists a unique $\mathcal{G}_{\tilde{R}_{\infty}}$ -equivariant ring endomorphism φ on $\hat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}}$ lifting the p -th power map on $\mathcal{O}_{\mathcal{E}}^{\text{ur}}/(p)$ and extending φ on $\mathcal{O}_{\mathcal{E}}$. Moreover, this map restricts to $\varphi : \mathfrak{S}^{\text{ur}} \rightarrow \mathfrak{S}^{\text{ur}}$, and the natural inclusion $\hat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}} \hookrightarrow W(\bar{R}^b[\frac{1}{\varpi}])$ is φ -equivariant.*

Proof. By [17, Lemma 7.2.7], we only need to prove the statement about the restriction of φ to \mathfrak{S}^{ur} . This follows from $\mathfrak{S}^{\text{ur}} = \hat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}} \cap W(\bar{R}^b)$ and that the inclusion $\hat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}} \hookrightarrow W(\bar{R}^b[\frac{1}{\varpi}])$ is φ -equivariant. \square

Let $\text{Rep}_{\mathbb{Z}_p}(\mathcal{G}_{\tilde{R}_{\infty}})$ be the category of finite \mathbb{Z}_p -modules equipped with continuous $\mathcal{G}_{\tilde{R}_{\infty}}$ -action. Let $\text{Rep}_{\mathbb{Z}_p}^{\text{free}}(\mathcal{G}_{\tilde{R}_{\infty}})$ and $\text{Rep}_{\mathbb{Z}_p}^{\text{tor}}(\mathcal{G}_{\tilde{R}_{\infty}})$ respectively denote the full subcategories of free and torsion objects. For $M \in \text{Mod}_{\mathcal{O}_{\mathcal{E}}}^{\text{ét}}(\varphi)$ and $T \in \text{Rep}_{\mathbb{Z}_p}(\mathcal{G}_{\tilde{R}_{\infty}})$, let $T(M) := (\hat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}} \otimes_{\mathcal{O}_{\mathcal{E}}} M)^{\varphi=1}$ and $D(T) := (\hat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}} \otimes_{\mathbb{Z}_p} T)^{\mathcal{G}_{\tilde{R}_{\infty}}}$. The following is proved in [17].

Proposition 4.20. (cf. [17, Proposition 7.3], [16, Lemma 4.1.1], [1, Theorem 7.11]) *The constructions T and D give exact quasi-inverse equivalences of \otimes -categories between $\text{Mod}_{\mathcal{O}_{\mathcal{E}}}^{\text{ét}}(\varphi)$ and $\text{Rep}_{\mathbb{Z}_p}(\mathcal{G}_{\tilde{R}_{\infty}})$. Moreover, T and D restrict to rank-preserving equivalences*

of categories between $\mathrm{Mod}_{\mathcal{O}_{\mathcal{E}}}^{\mathrm{\acute{e}t}, \mathrm{pr}}(\varphi)$ and $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{free}}(\mathcal{G}_{\tilde{R}_{\infty}})$, and length-preserving equivalences of categories between $\mathrm{Mod}_{\mathcal{O}_{\mathcal{E}}}^{\mathrm{\acute{e}t}, \mathrm{tor}}(\varphi)$ and $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{tor}}(\mathcal{G}_{\tilde{R}_{\infty}})$. In both cases, T and D commute with taking dual.

For M in $\mathrm{Mod}_{\mathcal{O}_{\mathcal{E}}}^{\mathrm{\acute{e}t}, \mathrm{pr}}(\varphi)$ (resp. $\mathrm{Mod}_{\mathcal{O}_{\mathcal{E}}}^{\mathrm{\acute{e}t}, \mathrm{tor}}(\varphi)$), we define the contravariant functor $T^{\vee}(M) := \mathrm{Hom}_{\mathcal{O}_{\mathcal{E}}, \varphi}(M, \hat{\mathcal{O}}_{\mathcal{E}}^{\mathrm{ur}})$ (resp. $\mathrm{Hom}_{\mathcal{O}_{\mathcal{E}}, \varphi}(M, \hat{\mathcal{O}}_{\mathcal{E}}^{\mathrm{ur}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)$). For a Kisin module \mathfrak{M} of height r , we define $T_{\mathfrak{S}}^{\vee}(\mathfrak{M}) := \mathrm{Hom}_{\mathfrak{S}, \varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}})$. For a torsion Kisin module \mathfrak{M} of height r , let $T_{\mathfrak{S}}^{\vee}(\mathfrak{M}) := \mathrm{Hom}_{\mathfrak{S}, \varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)$. Note that if we have a short exact sequence of étale $(\varphi, \mathcal{O}_{\mathcal{E}})$ -modules $0 \rightarrow \tilde{M}' \rightarrow \tilde{M} \rightarrow M \rightarrow 0$ where \tilde{M}' and \tilde{M} are $\mathcal{O}_{\mathcal{E}}$ -projective and M is p -power torsion, then we have a natural $\mathcal{G}_{\tilde{R}_{\infty}}$ -equivariant short exact sequence

$$0 \rightarrow T^{\vee}(\tilde{M}) \rightarrow T^{\vee}(\tilde{M}') \rightarrow T^{\vee}(M) \rightarrow 0.$$

Remark 4.21. Our notation for the contravariant functor $T_{\mathfrak{S}}^{\vee}$ is different from the one used in [25] for the case $R = \mathcal{O}_K$. In [25], the same functor is denoted by $T_{\mathfrak{S}}$.

Now, if G_R is a p -divisible group over R , we write $T_p(G_R) := \mathrm{Hom}_{\bar{R}}(\mathbb{Q}_p/\mathbb{Z}_p, G_R \times_R \bar{R})$, which is a finite free \mathbb{Z}_p -representation of \mathcal{G}_R . By [17, Corollary 8.2], we have a natural $\mathcal{G}_{\tilde{R}_{\infty}}$ -equivariant isomorphism $T^{\vee}(\mathfrak{M}^*(G_R) \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}) \cong T_p(G_R)$. If H is a p -power order finite locally free group scheme over R , then $H(\bar{R})$ is a finite torsion \mathbb{Z}_p -module equipped with a continuous action of \mathcal{G}_R . By [17, Proposition 9.5.1], there exists a natural $\mathcal{G}_{\tilde{R}_{\infty}}$ -equivariant isomorphism $H(\bar{R}) \cong T^{\vee}(\mathfrak{M}^*(H) \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}})$, and if $H = \ker(d : G^0 \rightarrow G^1)$ for some isogeny d of p -divisible groups, then the isomorphism $H(\bar{R}) \cong T^{\vee}(\mathfrak{M}^*(H) \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}})$ is compatible with the isomorphism $T_p(G^i) \cong T^{\vee}(\mathfrak{M}^*(G^i) \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}})$.

Note that any p -divisible group over $R[\frac{1}{p}]$ is étale, so the category of p -divisible groups over $R[\frac{1}{p}]$ is equivalent to the category of finite free \mathbb{Z}_p -representations of \mathcal{G}_R . Thus, by Theorem 4.10, Proposition 4.13 and 4.20, we see that for a p -divisible group G given as in

Question 4.2, the corresponding étale $(\varphi, \mathcal{O}_{\mathcal{E}})$ -module M satisfies the assumptions given in the following question.

Question 4.22. Let $M \in \text{Mod}_{\mathcal{O}_{\mathcal{E}}}^{\text{ét}, \text{pr}}(\varphi)$ such that for each $n \geq 1$, there exists a torsion Kisin module \mathfrak{M}_n of height 1 which is in the essential image of the functor \mathfrak{M}^* as in Proposition 4.13, and satisfies $\mathfrak{M}_n \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} \cong M/p^n$ as étale $(\varphi, \mathcal{O}_{\mathcal{E}})$ -modules. Then, does there exist a Kisin module $\mathfrak{M} \in \text{Mod}_{\mathfrak{S}}^1(\varphi, \nabla)$ such that $\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} \cong M$ as étale $(\varphi, \mathcal{O}_{\mathcal{E}})$ -modules?

We study Question 4.22 in the next sections for the cases $R = \mathcal{O}_K[[t]]$ and $R = \mathcal{O}_K\langle t^{\pm 1} \rangle$.

Remark 4.23. Note that studying Question 4.22 is sufficient to answer Question 4.2 when the natural forgetful functor from the category of \mathcal{G}_R -representations arising from p -divisible groups over R to the category $\text{Rep}_{\mathbb{Z}_p}^{\text{free}}(\mathcal{G}_{\tilde{R}_{\infty}})$ is fully faithful. This holds when $R = \mathcal{O}_K$ by [18, Corollary 2.1.14], but we do not know this for general R . However, for the base rings we consider (e.g. $R = \mathcal{O}_K[[t_1, \dots, t_d]]$ or $R = \mathcal{O}_K\langle t_1^{\pm 1}, \dots, t_d^{\pm 1} \rangle$), we show by direct constructions that studying Question 4.22 is sufficient to prove the theorems stated in the introduction.

4.3 $R = \mathcal{O}_K[[t]]$ Case

4.3.1 Weak Full Faithfulness of $T_{\mathfrak{S}}^{\vee}$

In this subsection, we let $R_0 = W(k)[[t]]$ so that $R = \mathcal{O}_K[[t]]$. We fix the Frobenius φ on R_0 to be given by $t \mapsto t^p$. In [25, Theorem 2.4.2], certain weak full faithfulness of the functor $T_{\mathfrak{S}}^{\vee}$ on the category of torsion Kisin modules of height r is shown when $R = \mathcal{O}_K$. We generalize this result to the case $R = \mathcal{O}_K[[t]]$. Let R'_0 be the p -adic completion of $\varinjlim_{\varphi} (R_0)_{(p)}$. By the universal property of Witt vectors, we have a φ -equivariant isomorphism $R'_0 \cong W(k((t))^{\text{perf}})$

where $k((t))^{\text{perf}} := \varinjlim_{\varphi} k((t)) = \varinjlim_{\varphi} \text{Frac}(R_0/p)$ denotes the perfect closure of $k((t))$. The natural φ -equivariant map $b_g : R_0 \rightarrow R'_0$ induces the base change

$$b_g : R \rightarrow R' := R'_0 \otimes_{W(k)} \mathcal{O}_K.$$

Let $\mathfrak{S}' := R'_0[[u]]$, and let $\mathcal{O}_{\mathcal{E}'}$ be the p -adic completion of $\mathfrak{S}'[\frac{1}{u}]$ and $\mathcal{E}' := \mathcal{O}_{\mathcal{E}'}[\frac{1}{p}]$. We similarly denote by \bar{R}' , \bar{R}'^b , $\mathfrak{S}'^{\text{ur}}$ and $\hat{\mathcal{O}}_{\mathcal{E}'}^{\text{ur}}$ the corresponding rings constructed for R' as in Section 4.2.2 compatibly with the base change b_g .

Lemma 4.24. *We have*

$$\mathcal{E} \cap \mathcal{O}_{\mathcal{E}'} = \mathcal{O}_{\mathcal{E}}, \quad \mathfrak{S}'^{\text{ur}} \cap \hat{\mathcal{O}}_{\mathcal{E}'}^{\text{ur}} = \mathfrak{S}^{\text{ur}}, \quad \text{and} \quad \mathfrak{S}^{\text{ur}}[\frac{1}{p}] \cap \hat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}} = \mathfrak{S}^{\text{ur}}$$

where all the intersections are taken inside $W(\bar{R}^b[\frac{1}{\varpi}][\frac{1}{p}])$.

Proof. Note that $\bar{R}^b[\frac{1}{\varpi}]$ is a perfectoid field which is the tilt of $\hat{\bar{R}}'[\frac{1}{p}]$, and it is equipped with the rank-1 valuation induced from that on $\hat{\bar{R}}'[\frac{1}{p}]$ satisfying $|\varpi|_{\bar{R}^b[\frac{1}{\varpi}]} = |\varpi|_{\hat{\bar{R}}'[\frac{1}{p}]}$ (cf. [34, Lemma 3.4]). If we regard $\bar{R}^b[\frac{1}{\varpi}]$ as a subring of $\bar{R}'^b[\frac{1}{\varpi}]$, then for any $x \in \bar{R}^b[\frac{1}{\varpi}]$, we have $x \in \bar{R}^b$ if and only if $|x|_{\bar{R}^b[\frac{1}{\varpi}]} \leq 1$. Thus, $\bar{R}^b = \bar{R}'^b[\frac{1}{\varpi}] \cap \bar{R}^b$. Every element $y \in W(\bar{R}^b[\frac{1}{\varpi}])$ is uniquely written as $y = \sum_{n=0}^{\infty} p^n [\bar{y}_n]$ with $\bar{y}_n \in \bar{R}'^b[\frac{1}{\varpi}]$ and $[\bar{y}_n]$ denoting the Teichmüller lift. So $W(\bar{R}^b) = W(\bar{R}'^b[\frac{1}{\varpi}]) \cap W(\bar{R}^b)$ inside $W(\bar{R}'^b[\frac{1}{\varpi}])$, and

$$\mathfrak{S}'^{\text{ur}} \cap \hat{\mathcal{O}}_{\mathcal{E}'}^{\text{ur}} = (W(\bar{R}'^b) \cap \hat{\mathcal{O}}_{\mathcal{E}'}^{\text{ur}}) \cap \hat{\mathcal{O}}_{\mathcal{E}'}^{\text{ur}} = W(\bar{R}'^b) \cap \hat{\mathcal{O}}_{\mathcal{E}'}^{\text{ur}} = \mathfrak{S}^{\text{ur}}$$

since $\hat{\mathcal{O}}_{\mathcal{E}'}^{\text{ur}} \subset W(\bar{R}'^b[\frac{1}{\varpi}])$.

Now, for any $x \in W(\bar{R}^b)[\frac{1}{p}]$, we have $x \in W(\bar{R}^b)$ if and only if the p -adic norm of $x \leq 1$. So $W(\bar{R}^b)[\frac{1}{p}] \cap W(\bar{R}'^b[\frac{1}{\varpi}]) = W(\bar{R}^b)$ inside $W(\bar{R}'^b[\frac{1}{\varpi}])[\frac{1}{p}]$. This implies $\mathfrak{S}^{\text{ur}}[\frac{1}{p}] \cap \hat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}} \subset W(\bar{R}^b)$, and therefore $\mathfrak{S}^{\text{ur}}[\frac{1}{p}] \cap \hat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}} = \mathfrak{S}^{\text{ur}}$.

The identity $\mathcal{E} \cap \mathcal{O}_{\mathcal{E}'} = \mathcal{O}_{\mathcal{E}}$ is clear by considering the p -adic norm as above. \square

Lemma 4.25. *For any $\mathfrak{M} \in \text{Mod}_{\mathfrak{S}}^{\text{tor}, r}(\varphi)$, there exist $\mathfrak{N}, \mathfrak{N}' \in \text{Mod}_{\mathfrak{S}}^r(\varphi)$ and a φ -equivariant exact sequence*

$$0 \rightarrow \mathfrak{N}' \rightarrow \mathfrak{N} \rightarrow \mathfrak{M} \rightarrow 0.$$

Proof. This can be proved by a similar argument as in the proof of [18, Lemma 2.3.4] as follows. Let $L := \text{coker}(1 \otimes \varphi_{\mathfrak{M}})$ and $Q := \text{im}(1 \otimes \varphi_{\mathfrak{M}})$. We choose a finite free $\mathfrak{S}/E(u)^r$ -module \tilde{L} which surjects onto L and a finite free \mathfrak{S} -module \mathfrak{N} which surjects onto both \mathfrak{M} and \tilde{L} . Set $\tilde{Q} := \ker(\mathfrak{N} \twoheadrightarrow \tilde{L})$. By replacing \mathfrak{N} by $\mathfrak{N} \oplus \mathfrak{S}^r$ where \mathfrak{S}^r surjects onto Q and maps to zero in \tilde{L} if necessary, we may assume that \tilde{Q} surjects onto Q under the projection $\mathfrak{N} \twoheadrightarrow \mathfrak{M}$. Then we have the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{Q} & \longrightarrow & \mathfrak{N} & \longrightarrow & \tilde{L} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Q & \longrightarrow & \mathfrak{M} & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

Since \tilde{L} is free over $\mathfrak{S}/E(u)^r$ and $\mathfrak{S}/E(u)^r$ has \mathfrak{S} -projective dimension 1, \tilde{Q} is free over \mathfrak{S} of the same rank as \mathfrak{N} . We may write $\tilde{Q} = \tilde{Q}_0 \oplus \tilde{Q}_1$ where \tilde{Q}_1 maps to 0 in Q and $\tilde{Q}_0 \otimes_{\mathfrak{S}} k \xrightarrow{\cong} Q \otimes_{\mathfrak{S}} k$. Note that $\varphi^*\mathfrak{M} \cong Q$ by Lemma 4.12. Since $\varphi^*\mathfrak{N}$ is a free \mathfrak{S} -module, the composite $\varphi^*\mathfrak{N} \twoheadrightarrow \varphi^*\mathfrak{M} \cong Q$ lifts to a map $\varphi^*\mathfrak{N} \rightarrow \tilde{Q}_0$, and any such lift is a surjection. We may then lift this further to a surjection $\varphi^*\mathfrak{N} \rightarrow \tilde{Q}$ and any such lift is an isomorphism, since $\varphi^*\mathfrak{N}$ and \tilde{Q} are free \mathfrak{S} -modules of the same rank. We choose such a lift $\varphi^*\mathfrak{N} \cong \tilde{Q}$, and define $\varphi_{\mathfrak{N}}$ by

$$1 \otimes \varphi_{\mathfrak{N}} : \varphi^*\mathfrak{N} \cong \tilde{Q} \hookrightarrow \mathfrak{N}.$$

Then $(\mathfrak{N}, \varphi_{\mathfrak{N}}) \in \text{Mod}_{\mathfrak{S}}^r(\varphi)$ and the projection $\mathfrak{N} \twoheadrightarrow \mathfrak{M}$ is φ -equivariant. Let $\mathfrak{N}' := \ker(\mathfrak{N} \twoheadrightarrow \mathfrak{M})$ with $\varphi_{\mathfrak{N}'}$ induced from $\varphi_{\mathfrak{N}}$. Note that \mathfrak{N}' is finite projective over \mathfrak{S} . Since $\varphi : \mathfrak{S} \rightarrow \mathfrak{S}$

is flat by [6, Lemma 7.1.8], we have the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \varphi^*(\mathfrak{N}') & \longrightarrow & \varphi^*(\mathfrak{N}) & \longrightarrow & \varphi^*(\mathfrak{M}) & \longrightarrow & 0 \\
& & \downarrow 1 \otimes \varphi_{\mathfrak{N}'} & & \downarrow 1 \otimes \varphi_{\mathfrak{N}} & & \downarrow 1 \otimes \varphi_{\mathfrak{M}} & & \\
0 & \longrightarrow & \mathfrak{N}' & \longrightarrow & \mathfrak{N} & \longrightarrow & \mathfrak{M} & \longrightarrow & 0
\end{array}$$

By the Snake Lemma, this induces a natural embedding $\text{coker}(1 \otimes \varphi_{\mathfrak{N}'}) \hookrightarrow \text{coker}(1 \otimes \varphi_{\mathfrak{N}})$ since $1 \otimes \varphi_{\mathfrak{M}}$ is injective. Thus, $\text{coker}(1 \otimes \varphi_{\mathfrak{N}'})$ is killed by $E(u)^r$, and $\mathfrak{N}' \in \text{Mod}_{\mathfrak{S}}^r(\varphi)$. \square

Lemma 4.26. *Let \mathfrak{M} be a (torsion) Kisin module of height r . Then the natural map $T_{\mathfrak{S}}^{\vee}(\mathfrak{M}) \rightarrow T^{\vee}(\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}})$ induced by the natural inclusion $\mathfrak{S}^{\text{ur}} \hookrightarrow \hat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}}$ is an isomorphism.*

Proof. First, suppose that \mathfrak{M} is a Kisin module of height r . Consider the base change $b_g : R \rightarrow R'$ as above. The natural morphism $T^{\vee}(\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}) \rightarrow T^{\vee}(\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}'})$ is an injective map of finite free \mathbb{Z}_p -modules of the same rank. Its image is saturated since $\mathcal{E} \cap \mathcal{O}_{\mathcal{E}'} = \mathcal{O}_{\mathcal{E}}$ in \mathcal{E}' by Lemma 4.24, so it is an isomorphism. Note that $\mathfrak{M}' := \mathfrak{M} \otimes_{\mathfrak{S}, b_g} \mathfrak{S}'$ is a Kisin module of height r over \mathfrak{S}' . Since $R' \cong W(k((t))^{\text{perf}}) \otimes_{W(k)} \mathcal{O}_K$, the natural map $T_{\mathfrak{S}'}^{\vee}(\mathfrak{M}') \rightarrow T^{\vee}(\mathfrak{M}' \otimes_{\mathfrak{S}'} \mathcal{O}_{\mathcal{E}'})$ is an isomorphism by [25, Corollary 2.2.2]. Thus, any φ -equivariant map from \mathfrak{M} to $\hat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}}$ has the image in $\mathfrak{S}'^{\text{ur}} \cap \hat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}} = \mathfrak{S}^{\text{ur}}$ by Lemma 4.24, and the map $T_{\mathfrak{S}}^{\vee}(\mathfrak{M}) \rightarrow T^{\vee}(\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}})$ is an isomorphism.

Now, suppose that \mathfrak{M} is a torsion Kisin module of height r . By Lemma 4.24, the natural map $\mathfrak{S}^{\text{ur}}[\frac{1}{p}]/\mathfrak{S}^{\text{ur}} \rightarrow \hat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}}[\frac{1}{p}]/\hat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}}$ is injective, and the map $T_{\mathfrak{S}}^{\vee}(\mathfrak{M}) \rightarrow T^{\vee}(\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}})$ is injective.

By Lemma 4.25, there exists a φ -equivariant exact sequence

$$0 \rightarrow \tilde{\mathfrak{N}} \rightarrow \mathfrak{N} \rightarrow \mathfrak{M} \rightarrow 0$$

for some Kisin modules $\tilde{\mathfrak{N}}$, \mathfrak{N} of height r . This yields an exact sequence of étale $(\varphi, \mathcal{O}_{\mathcal{E}})$ -

modules

$$0 \rightarrow \tilde{\mathfrak{N}} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} \rightarrow \mathfrak{N} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} \rightarrow \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} \rightarrow 0.$$

By Proposition 4.20, this induces the natural $\mathcal{G}_{\tilde{R}_{\infty}}$ -equivariant short exact sequence

$$0 \rightarrow T^{\vee}(\mathfrak{N} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}) \rightarrow T^{\vee}(\tilde{\mathfrak{N}} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}) \rightarrow T^{\vee}(\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}) \rightarrow 0.$$

The maps $T_{\mathfrak{S}}^{\vee}(\mathfrak{N}) \rightarrow T^{\vee}(\mathfrak{N} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}})$ and $T_{\mathfrak{S}}^{\vee}(\tilde{\mathfrak{N}}) \rightarrow T^{\vee}(\tilde{\mathfrak{N}} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}})$ are isomorphisms by above. Note that the natural map $T_{\mathfrak{S}}^{\vee}(\tilde{\mathfrak{N}}) \rightarrow T_{\mathfrak{S}}^{\vee}(\mathfrak{M})$ induced by the isomorphism $\tilde{\mathfrak{N}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\cong} \mathfrak{N} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ fits into the following commutative diagram.

$$\begin{array}{ccc} T_{\mathfrak{S}}^{\vee}(\tilde{\mathfrak{N}}) & \longrightarrow & T_{\mathfrak{S}}^{\vee}(\mathfrak{M}) \\ \downarrow \cong & & \downarrow \\ T^{\vee}(\tilde{\mathfrak{N}} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}) & \twoheadrightarrow & T^{\vee}(\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}) \end{array}$$

Since the left vertical map and bottom map are surjective, the right vertical map is surjective. Thus, the map $T_{\mathfrak{S}}^{\vee}(\mathfrak{M}) \rightarrow T^{\vee}(\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}})$ is an isomorphism. \square

Let $\mathfrak{S}_n := \mathfrak{S}/p^n$. For studying torsion Kisin modules of height r , it is convenient to work with a certain subring of $\mathfrak{S}_n^{\text{ur}} := \mathfrak{S}^{\text{ur}}/p^n$. We define $F_{\mathfrak{S}}^{f(r)}(\mathfrak{S}_n^{\text{ur}})$ to be the set of φ -stable finite \mathfrak{S}_n -submodules of $\mathfrak{S}_n^{\text{ur}}$ such that $\text{coker}(1 \otimes \varphi)$ is killed by $E(u)^r$. Let $\mathfrak{S}_n^{f(r)} := \bigcup_{\mathfrak{M} \in F_{\mathfrak{S}}^{f(r)}(\mathfrak{S}_n^{\text{ur}})} \mathfrak{M}$. Note that $\mathfrak{S}_n^{f(r)}$ is a φ -stable \mathfrak{S}_n -subalgebra of $\mathfrak{S}_n^{\text{ur}}$, and it is $\mathcal{G}_{\tilde{R}_{\infty}}$ -stable since φ and $\mathcal{G}_{\tilde{R}_{\infty}}$ -action on $\mathfrak{S}_n^{\text{ur}}$ commute. The following lemma is immediate.

Lemma 4.27. *For a torsion Kisin module \mathfrak{M} of height r which is killed by p^n , we have the natural isomorphism*

$$\text{Hom}_{\mathfrak{S}, \varphi}(\mathfrak{M}, \mathfrak{S}_n^{f(r)}) \cong \text{Hom}_{\mathfrak{S}, \varphi}(\mathfrak{M}, \mathfrak{S}_n^{\text{ur}}) = T_{\mathfrak{S}}^{\vee}(\mathfrak{M}).$$

Now, we study the duality for (torsion) Kisin modules of height r . We define a φ -semilinear morphism $\varphi^\vee : \mathfrak{S} \rightarrow \mathfrak{S}$ by $1 \mapsto c_0^{-r} E(u)^r$, where pc_0 is the constant term of $E(u)$. Denote by \mathfrak{S}^\vee the ring \mathfrak{S} equipped with φ^\vee . Similarly, we denote by \mathfrak{S}_n^\vee , $\mathfrak{S}_n^{f(r),\vee}$, $\mathfrak{S}_n^{\text{ur},\vee}$ and $\mathfrak{S}^{\text{ur},\vee}$ the corresponding rings equipped with φ^\vee . Suppose $r = 1$. From the construction in [17] of the functor \mathfrak{M}^* in Theorem 4.10 using Dieudonné crystals and by [4, Example 2.2.3], we see that $\mathfrak{M}^*(\mu_{p^\infty}) = \mathfrak{S}^\vee$, and therefore $\mathfrak{M}^*(\mu_{p^n}) = \mathfrak{S}_n^\vee$. Thus, by [17, Corollary 8.2, Proposition 9.5.1] and Lemma 4.26, we have isomorphisms of $\mathbb{Z}_p[\mathcal{G}_{\tilde{R}_\infty}]$ -modules

$$T_p(\mu_{p^\infty}) \cong \mathbb{Z}_p(1) \cong T^\vee(\mathfrak{S}^\vee \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}) \cong T_{\mathfrak{S}^\vee}(\mathfrak{S}^\vee)$$

and

$$T_p(\mu_{p^n}) \cong (\mathbb{Z}_p/p^n)(1) \cong T^\vee(\mathfrak{S}_n^\vee \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}) \cong T_{\mathfrak{S}^\vee}(\mathfrak{S}_n^\vee).$$

By the definition of the functor T in Proposition 4.20, this implies that for any positive integer r , we have $T_{\mathfrak{S}^\vee}(\mathfrak{S}^\vee) \cong \mathbb{Z}_p(r)$ and $T_{\mathfrak{S}^\vee}(\mathfrak{S}_n^\vee) \cong (\mathbb{Z}_p/p^n)(r)$ if φ^\vee is given by $1 \mapsto c_0^{-r} E(u)^r$.

For $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ a (torsion) Kisin module of height r , let $\psi_{\mathfrak{M}} : \mathfrak{M} \rightarrow \varphi^* \mathfrak{M}$ be the unique \mathfrak{S} -linear morphism as in Section 4.2.1 such that $(1 \otimes \varphi_{\mathfrak{M}}) \circ \psi_{\mathfrak{M}} = c_0^{-r} E(u)^r \text{Id}_{\mathfrak{M}}$ and $\psi_{\mathfrak{M}} \circ (1 \otimes \varphi_{\mathfrak{M}}) = c_0^{-r} E(u)^r \text{Id}_{\varphi^* \mathfrak{M}}$. Let $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ be a Kisin module of height r . We define the dual of \mathfrak{M} to be $\mathfrak{M}^\vee := \text{Hom}_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{S})$. We have a natural isomorphism $\varphi^*(\mathfrak{M}^\vee) \xrightarrow{\cong} \text{Hom}_{\mathfrak{S}}(\varphi^* \mathfrak{M}, \mathfrak{S})$. Let $\psi_{\mathfrak{M}}^\vee : \text{Hom}_{\mathfrak{S}}(\varphi^* \mathfrak{M}, \mathfrak{S}) \rightarrow \mathfrak{M}^\vee$ be the dual map of $\psi_{\mathfrak{M}}$. We equip \mathfrak{M}^\vee with the φ -semilinear endomorphism $\varphi_{\mathfrak{M}^\vee}$ given by the composite

$$1 \otimes \varphi_{\mathfrak{M}^\vee} : \varphi^*(\mathfrak{M}^\vee) \xrightarrow{\cong} \text{Hom}_{\mathfrak{S}}(\varphi^* \mathfrak{M}, \mathfrak{S}) \xrightarrow{\psi_{\mathfrak{M}}^\vee} \mathfrak{M}^\vee.$$

Then $(\mathfrak{M}^\vee, \varphi_{\mathfrak{M}^\vee})$ is a Kisin module of height r . We have a more explicit description of

$(\mathfrak{M}^\vee, \varphi_{\mathfrak{M}^\vee})$ as follows. Let (e_1, \dots, e_h) be a \mathfrak{S} -basis of \mathfrak{M} , and write $\varphi_{\mathfrak{M}}(e_1, \dots, e_h) = (e_1, \dots, e_h)A$ where A is a $h \times h$ matrix with coefficients in \mathfrak{S} . Then, if $(e_1^\vee, \dots, e_h^\vee)$ is the dual basis of \mathfrak{M}^\vee , we have

$$\varphi_{\mathfrak{M}^\vee}(e_1^\vee, \dots, e_h^\vee) = (e_1^\vee, \dots, e_h^\vee)(c_0^{-r} E(u)^r)(A^{-1})^t.$$

Note that $(c_0^{-r} E(u)^r)(A^{-1})^t$ has coefficients in \mathfrak{S} since $\text{coker}(1 \otimes \varphi_{\mathfrak{M}})$ is killed by $E(u)^r$. From this description and Lemma 4.26, we see that $T_{\mathfrak{S}}^\vee(\mathfrak{M}^\vee) \cong T_{\mathfrak{S}}(\mathfrak{M})(r)$ as $\mathbb{Z}_p[\mathcal{G}_{\hat{R}_\infty}]$ -modules, where we define $T_{\mathfrak{S}}(\mathfrak{M}) := \text{Hom}_{\mathbb{Z}_p}(T_{\mathfrak{S}}^\vee(\mathfrak{M}), \mathbb{Z}_p)$ as a functor on Kisin modules.

If $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ is a torsion Kisin module of height r , then we define its dual to be $\mathfrak{M}^\vee := \text{Ext}_{\mathfrak{S}}^1(\mathfrak{M}, \mathfrak{S})$. By Lemma 4.25, there is a φ -equivariant exact sequence

$$0 \rightarrow \mathfrak{N}' \rightarrow \mathfrak{N} \rightarrow \mathfrak{M} \rightarrow 0$$

for some Kisin modules $\mathfrak{N}, \mathfrak{N}'$ of height r . Since $\varphi : \mathfrak{S} \rightarrow \mathfrak{S}$ is flat by [6, Lemma 7.1.8], we have the following commutative diagram with exact rows.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \varphi^*(\mathfrak{N}^\vee) & \longrightarrow & \varphi^*((\mathfrak{N}')^\vee) & \longrightarrow & \varphi^*(\mathfrak{M}^\vee) \longrightarrow 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \\ 0 & \longrightarrow & \text{Hom}_{\mathfrak{S}}(\varphi^*\mathfrak{N}, \mathfrak{S}) & \longrightarrow & \text{Hom}_{\mathfrak{S}}(\varphi^*\mathfrak{N}', \mathfrak{S}) & \longrightarrow & \text{Ext}_{\mathfrak{S}}^1(\varphi^*\mathfrak{M}, \mathfrak{S}) \longrightarrow 0 \end{array}$$

Since the left and middle vertical map are isomorphisms, the right vertical map is an isomorphism. We equip \mathfrak{M}^\vee with the φ -semilinear map $\varphi_{\mathfrak{M}^\vee}$ given by the composite

$$1 \otimes \varphi_{\mathfrak{M}^\vee} : \varphi^*(\mathfrak{M}^\vee) \xrightarrow{\cong} \text{Ext}_{\mathfrak{S}}^1(\varphi^*\mathfrak{M}, \mathfrak{S}) \xrightarrow{\psi_{\mathfrak{M}}^\vee} \mathfrak{M}^\vee,$$

where $\psi_{\mathfrak{M}}^\vee : \text{Ext}_{\mathfrak{S}}^1(\varphi^*\mathfrak{M}, \mathfrak{S}) \rightarrow \mathfrak{M}^\vee$ is the map induced by $\psi_{\mathfrak{M}}$. Then $(\mathfrak{M}^\vee, \varphi_{\mathfrak{M}^\vee})$ is a torsion

Kisin module of height r . Furthermore, we have the exact sequence of $\mathbb{Z}_p[\mathcal{G}_{\tilde{R}_\infty}]$ -modules

$$0 \rightarrow T_{\mathfrak{S}}^{\vee}(\mathfrak{N}^{\vee}) \rightarrow T_{\mathfrak{S}}^{\vee}((\mathfrak{N}')^{\vee}) \rightarrow T_{\mathfrak{S}}^{\vee}(\mathfrak{M}^{\vee}) \rightarrow 0.$$

Since $T_{\mathfrak{S}}^{\vee}(\mathfrak{N}^{\vee}) \cong T_{\mathfrak{S}}(\mathfrak{N})(r)$ and $T_{\mathfrak{S}}^{\vee}((\mathfrak{N}')^{\vee}) \cong T_{\mathfrak{S}}(\mathfrak{N}')(r)$, we have $T_{\mathfrak{S}}^{\vee}(\mathfrak{M}^{\vee}) \cong T_{\mathfrak{S}}(\mathfrak{M})(r)$, where we define $T_{\mathfrak{S}}(\mathfrak{M}) := \text{Hom}_{\mathbb{Z}_p}(T_{\mathfrak{S}}^{\vee}(\mathfrak{M}), \mathbb{Q}_p/\mathbb{Z}_p)$ as a functor on torsion Kisin modules. Note that if $\mathfrak{M} \cong \bigoplus_{i \in I} \mathfrak{S}/p^{n_i}$, then $\mathfrak{M}^{\vee} \cong \text{Hom}_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{S}[\frac{1}{p}]/\mathfrak{S}) \cong \text{Hom}_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{S}_n)$ where $n = \max_{i \in I} n_i$. The following lemma is straightforward.

Lemma 4.28. *Let $0 \rightarrow \mathfrak{M} \rightarrow \mathfrak{N} \rightarrow \mathfrak{L} \rightarrow 0$ be an exact sequence in $\text{Mod}_{\mathfrak{S}}^{\text{tor}, r}(\varphi)$. Then the induced sequence $0 \rightarrow \mathfrak{L}^{\vee} \rightarrow \mathfrak{N}^{\vee} \rightarrow \mathfrak{M}^{\vee} \rightarrow 0$ is exact in $\text{Mod}_{\mathfrak{S}}^{\text{tor}, r}(\varphi)$.*

For torsion Kisin modules of height r , we construct morphisms to compare \mathfrak{M} with $T_{\mathfrak{S}}(\mathfrak{M})$ similarly as in [25, Section 3.1].

Proposition 4.29. (cf. [25, Proposition 3.2.1]) *Let $\mathfrak{M} \in \text{Mod}_{\mathfrak{S}}^{\text{tor}, r}(\varphi)$. There exists a natural \mathfrak{S}^{ur} -linear morphism $\hat{\iota} : \mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}^{\text{ur}} \rightarrow T_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathfrak{S}^{\text{ur}}$ which is $\mathcal{G}_{\tilde{R}_\infty}$ -equivariant and φ -equivariant such that $\hat{\iota} \otimes_{\mathfrak{S}^{\text{ur}}} \hat{\mathcal{O}}_{\mathfrak{E}}^{\text{ur}}$ is an isomorphism.*

Proof. Given Lemma 4.26, it follows from the same argument as in the proof of [25, Proposition 3.2.1]:

Suppose \mathfrak{M} is killed by p^n . Note that $T_{\mathfrak{S}}^{\vee}(\mathfrak{M}) = \text{Hom}_{\mathfrak{S}^{\text{ur}}, \varphi}(\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}^{\text{ur}}, \mathfrak{S}_n^{\text{ur}})$. For each $f \in \text{Hom}_{\mathfrak{S}^{\text{ur}}, \varphi}(\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}^{\text{ur}}, \mathfrak{S}_n^{\text{ur}})$, the $\mathcal{G}_{\tilde{R}_\infty}$ -action on f is defined by $f^{\sigma}(m) = \sigma(f(\sigma^{-1}(m \otimes s)))$ for any $\sigma \in \mathcal{G}_{\tilde{R}_\infty}$ and $m \otimes s \in \mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}^{\text{ur}}$. We define a natural morphism $\hat{\iota}' : \mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}^{\text{ur}} \rightarrow \text{Hom}_{\mathbb{Z}_p}(T_{\mathfrak{S}}^{\vee}(\mathfrak{M}), \mathfrak{S}_n^{\text{ur}})$ by

$$m \otimes s \mapsto (f \mapsto f(m \otimes s), \forall f \in T_{\mathfrak{S}}^{\vee}(\mathfrak{M})).$$

On the other hand, since $T_{\mathfrak{S}}^{\vee}(\mathfrak{M}) \cong \bigoplus_i \mathbb{Z}_p/p^{n_i}$ as finite \mathbb{Z}_p -modules, we have a natural isomorphism $h : \text{Hom}_{\mathbb{Z}_p}(T_{\mathfrak{S}}^{\vee}(\mathfrak{M}), \mathfrak{S}_n^{\text{ur}}) \cong T_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathfrak{S}^{\text{ur}}$. This gives a natural morphism $\hat{i} := h \circ \hat{i}' : \mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}^{\text{ur}} \rightarrow T_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathfrak{S}^{\text{ur}}$. From the definitions, it follows immediately that \hat{i} is $\mathcal{G}_{\tilde{R}_{\infty}}$ -equivariant and φ -equivariant. By Proposition 4.20 and Lemma 4.26, $\hat{i} \otimes_{\mathfrak{S}^{\text{ur}}} \hat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}}$ is an isomorphism. \square

By Proposition 4.29, we have the $\mathfrak{S}_n^{\text{ur}}$ -linear morphism

$$\hat{i} : \mathfrak{S}_n^{\text{ur}, \vee} \rightarrow \mathfrak{S}_n^{\text{ur}}(-r)$$

with $\hat{i}(1) = \mathfrak{t}^r$. Here, $\mathfrak{t} \in \mathfrak{S}^{\text{ur}}$ satisfies $\varphi(\mathfrak{t}) = c_0^{-1}E(u)\mathfrak{t}$. Note that \mathfrak{t} is unique up to multiplication by a unit in \mathbb{Z}_p . When $\mathfrak{M} \cong \bigoplus_i \mathfrak{S}/p^{n_i}$, we further have the following construction.

Proposition 4.30. (cf. [25, Theorem 3.2.2]) *Let $\mathfrak{M} \in \text{Mod}_{\mathfrak{S}^{\text{tor}, r}}(\varphi)$ such that $\mathfrak{M} \cong \bigoplus_{i \in I} \mathfrak{S}/p^{n_i}$, and let $n = \max_{i \in I} n_i$. Then, there exists a natural \mathfrak{S}^{ur} -linear morphism*

$$\hat{i}^{\vee} : T_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathfrak{S}^{\text{ur}, \vee} \rightarrow \mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}^{\text{ur}}(-r)$$

which is $\mathcal{G}_{\tilde{R}_{\infty}}$ -equivariant and φ -equivariant. If we identify \mathfrak{S}^{ur} with $\mathfrak{S}^{\text{ur}, \vee}$ by ignoring the φ -structures, then $\hat{i}^{\vee} \circ \hat{i} = \text{Id} \otimes_{\mathfrak{S}} \mathfrak{t}^r$.

Furthermore, restricting \hat{i}^{\vee} to $T_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathfrak{S}_n^{f(r)}$ gives a natural injection

$$\hat{i}^{\vee} : T_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathfrak{S}_n^{f(r)} \rightarrow \mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}_n^{f(2r)}(-r).$$

Proof. Since $\mathfrak{M} \cong \bigoplus_{i \in I} \mathfrak{S}/p^{n_i}$, we have $\mathfrak{M}^\vee \cong \text{Hom}_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{S}_n)$, and the natural morphisms

$$\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}_n^{f(r)} \rightarrow \mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}_n^{\text{ur}} \rightarrow (\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}) \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}_{\mathcal{E},n}^{\text{ur}}$$

are injections (the natural map $\mathfrak{S}_n^{\text{ur}} \rightarrow \mathcal{O}_{\mathcal{E},n}^{\text{ur}}$ is injective since $\mathfrak{S}^{\text{ur}}[\frac{1}{p}]/\mathfrak{S}^{\text{ur}} \rightarrow \hat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}}[\frac{1}{p}]/\hat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}}$ is injective). Given these, it follows from the same argument as in the proof of [25, Theorem 3.2.2]:

First, we claim that there exists a natural isomorphism

$$\text{Hom}_{\mathfrak{S},\varphi}(\mathfrak{S}^\vee, \mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}_n^{f(r)}) \xrightarrow{\cong} \text{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}^\vee, \mathfrak{S}_n^{f(r)}) = T_{\mathfrak{S}}(\mathfrak{M})(r).$$

Indeed, it suffices by duality to construct a natural isomorphism

$$\text{Hom}_{\mathfrak{S},\varphi}(\mathfrak{S}^\vee, \mathfrak{M}^\vee \otimes_{\mathfrak{S}} \mathfrak{S}_n^{f(r)}) \xrightarrow{\cong} \text{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}_n^{f(r)}). \quad (4.1)$$

If we ignore φ -structures, then we have natural isomorphisms

$$\text{Hom}_{\mathfrak{S}}(\mathfrak{S}, \mathfrak{M}^\vee \otimes_{\mathfrak{S}} \mathfrak{S}_n^{f(r)}) \xrightarrow{\cong} \mathfrak{M}^\vee \otimes_{\mathfrak{S}} \mathfrak{S}_n^{f(r)} \xrightarrow{\cong} \text{Hom}_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{S}_n^{f(r)})$$

since $\mathfrak{M} \cong \bigoplus_{i \in I} \mathfrak{S}/p^{n_i}$. It remains to check that the φ -structures on both sides of (4.1) cut out the same elements under the given isomorphism. For $f \in \text{Hom}_{\mathfrak{S},\varphi}(\mathfrak{S}^\vee, \mathfrak{M}^\vee \otimes_{\mathfrak{S}} \mathfrak{S}_n^{f(r)})$, write $f(1) = \sum_j f_j \otimes a_j$ with $f_j \in \mathfrak{M}^\vee$ and $a_j \in \mathfrak{S}_n^{f(r)}$. We have

$$\varphi_{\mathfrak{M}^\vee \otimes_{\mathfrak{S}} \mathfrak{S}_n^{f(r)}}(f(1)) = f(\varphi_{\mathfrak{S}^\vee}(1)) = f(c_0^{-r} E(u)^r) = c_0^{-r} E(u)^r f(1).$$

Let $h = \sum_j a_j f_j \in \text{Hom}_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{S}_n^{f(r)})$. We have $c_0^{-r} E(u)^r \sum_j a_j f_j = \sum_j \varphi(a_j) \varphi_{\mathfrak{M}^\vee}(f_j)$.

Thus, for any $m \in \mathfrak{M}$,

$$c_0^{-r} E(u)^r h(\varphi_{\mathfrak{M}}(m)) = \sum_j \varphi(a_j) \varphi_{\mathfrak{M}^\vee}(f_j)(\varphi_{\mathfrak{M}}(m)).$$

Since $\varphi_{\mathfrak{M}^\vee}(f_j)(\varphi_{\mathfrak{M}}(m)) = c_0^{-r} E(u)^r \varphi(f_j(m))$, we have

$c_0^{-r} E(u)^r h(\varphi_{\mathfrak{M}}(m)) = c_0^{-r} E(u)^r \varphi(h(m))$, and therefore $h(\varphi_{\mathfrak{M}}(m)) = \varphi(h(m))$ since $E(u)$ is not a zero divisor in \mathfrak{S}_n . This proves the existence of the natural isomorphism (4.1).

Now, let $M = \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$. By Proposition 4.29, we have an $\hat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}}$ -linear isomorphism $\mathfrak{t}^r \otimes_{\mathfrak{S}^{\text{ur}}} \hat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}} : \mathcal{O}_{\mathcal{E},n}^{\text{ur},\vee} \cong \mathcal{O}_{\mathcal{E},n}^{\text{ur}}(-r)$. Thus, $\text{Hom}_{\hat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}},\varphi}(\hat{\mathcal{O}}_{\mathcal{E}}^{\text{ur},\vee}, M \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}_{\mathcal{E},n}^{\text{ur}}) = T^{\vee}(M)(r)$. Consider the natural maps

$$\text{Hom}_{\mathfrak{S},\varphi}(\mathfrak{S}^{\vee}, \mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}_n^{f(r)}) \rightarrow \text{Hom}_{\mathfrak{S}^{\text{ur}},\varphi}(\mathfrak{S}^{\text{ur},\vee}, \mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}_n^{\text{ur}}) \rightarrow \text{Hom}_{\hat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}},\varphi}(\hat{\mathcal{O}}_{\mathcal{E}}^{\text{ur},\vee}, M \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}_{\mathcal{E},n}^{\text{ur}}).$$

These maps are injective since the maps

$$\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}_n^{f(r)} \rightarrow \mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}_n^{\text{ur}} \rightarrow (\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}) \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}_{\mathcal{E},n}^{\text{ur}}$$

are injective. Since

$$\text{Hom}_{\mathfrak{S},\varphi}(\mathfrak{S}^{\vee}, \mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}_n^{f(r)}) \cong T_{\mathfrak{S}}(\mathfrak{M})(r) \cong \text{Hom}_{\hat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}},\varphi}(\hat{\mathcal{O}}_{\mathcal{E}}^{\text{ur},\vee}, M \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}_{\mathcal{E},n}^{\text{ur}})$$

as finite \mathbb{Z}_p -modules, the natural map

$$\text{Hom}_{\mathfrak{S}^{\text{ur}},\varphi}(\mathfrak{S}^{\text{ur},\vee}, \mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}_n^{\text{ur}}) \rightarrow \text{Hom}_{\hat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}},\varphi}(\hat{\mathcal{O}}_{\mathcal{E}}^{\text{ur},\vee}, M \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}_{\mathcal{E},n}^{\text{ur}})$$

is an isomorphism. By the same reasoning as in the proof of Proposition 4.29, we see that

there exists a natural φ -equivariant, $\mathcal{G}_{\tilde{R}_\infty}$ -equivariant and \mathfrak{S}^{ur} -linear morphism

$$\hat{\iota}^\vee : T_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathfrak{S}^{\text{ur},\vee} \rightarrow \mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}^{\text{ur}}(-r).$$

We claim that $\hat{\iota}^\vee \circ \hat{\iota} = \text{Id} \otimes_{\mathfrak{S}} \mathfrak{t}^r$. For this, it suffices to check that

$$(\hat{\iota}^\vee \otimes_{\mathfrak{S}^{\text{ur}}} \hat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}}) \circ (\hat{\iota} \otimes_{\mathfrak{S}^{\text{ur}}} \hat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}}) = \text{Id}_M \otimes_{\mathcal{O}_{\mathcal{E}}} (\mathfrak{t}^r \otimes_{\mathfrak{S}^{\text{ur}}} \hat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}}).$$

Note that $M \cong \bigoplus_{i \in I} \mathcal{O}_{\mathcal{E}}/p^{n_i}$ as $\mathcal{O}_{\mathcal{E}}$ -modules and $T^\vee(M) \cong \bigoplus_{i \in I} \mathbb{Z}_p/p^{n_i}$ as \mathbb{Z}_p -modules.

Thus, it suffices to show that

$$(\hat{\iota} \otimes_{\mathfrak{S}^{\text{ur}}} \hat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}}) \circ (\hat{\iota}^\vee \otimes_{\mathfrak{S}^{\text{ur}}} \hat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}}) = \text{Id}_{T^\vee(M)} \otimes_{\mathbb{Z}_p} (\mathfrak{t}^r \otimes_{\mathfrak{S}^{\text{ur}}} \hat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}}),$$

and this is clear from above constructions.

It remains to show the last statement about the restriction of $\hat{\iota}^\vee$. Since

$\hat{\iota}^\vee(T_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathfrak{S}^\vee) \subset \mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}_n^{f(r)}(-r)$ and $\hat{\iota}^\vee$ is \mathfrak{S}^{ur} -linear, it suffices to check that $\mathfrak{S}_n^{f(r)} \cdot \mathfrak{S}_n^{f(r)} \subset \mathfrak{S}_n^{f(2r)}$ inside $\mathfrak{S}_n^{\text{ur}}$. Let $\mathfrak{N}_1, \mathfrak{N}_2 \in F_{\mathfrak{S}}^{f(r)}(\mathfrak{S}_n^{\text{ur}})$, and let \mathfrak{N} be the \mathfrak{S} -submodule of $\mathfrak{S}_n^{\text{ur}}$ generated by $\mathfrak{N}_1 \cdot \mathfrak{N}_2$. For any $x \in \mathfrak{N}_1$ and $y \in \mathfrak{N}_2$, we have $E(u)^r x = \sum_j a_j \varphi(x_j)$ and $E(u)^r y = \sum_l b_l \varphi(y_l)$ for some $a_j, b_l \in \mathfrak{S}$ and $x_j \in \mathfrak{N}_1$, $y_l \in \mathfrak{N}_2$. Thus, $E(u)^{2r} xy = \sum_{j,l} a_j b_l \varphi(x_j y_l)$, and $\mathfrak{N} \in F_{\mathfrak{S}}^{f(2r)}(\mathfrak{S}_n^{\text{ur}})$. \square

We now state the weak full faithfulness of the functor $T_{\mathfrak{S}}^\vee$.

Theorem 4.31. (generalizing [25, Theorem 2.4.2]) *Let $\mathfrak{M}, \mathfrak{M}' \in \text{Mod}_{\mathfrak{S}}^{\text{tor},r}(\varphi)$ and $f : T_{\mathfrak{S}}^\vee(\mathfrak{M}') \rightarrow T_{\mathfrak{S}}^\vee(\mathfrak{M})$ be a morphism of $\mathbb{Z}_p[\mathcal{G}_{\tilde{R}_\infty}]$ -modules. Then there exists a morphism of torsion Kisin modules $\mathfrak{f} : \mathfrak{M} \rightarrow \mathfrak{M}'$ such that $T_{\mathfrak{S}}^\vee(\mathfrak{f}) = p^{\mathfrak{c}} f$, where \mathfrak{c} is a constant depending only on e and r . We have $\mathfrak{c} = 0$ if $er < p - 1$.*

To prove Theorem 4.31, we first reduce to the case when \mathfrak{M} is a free \mathfrak{S}_n -module of rank 1 similarly as in [25, Section 4]. The main difference to the case $R = \mathcal{O}_K$ is that the “scheme theoretic closure” as given in [25, Lemma 2.3.6] does not yield a torsion Kisin module of height r in general (cf. Remark 4.46 for the case $r = 1$). Instead, we study the following natural construction.

Lemma 4.32. *Let \mathfrak{M} be a finite u -torsion free \mathfrak{S}_n -module equipped with a φ -semilinear endomorphism $\varphi_{\mathfrak{M}}$ such that $\text{coker}(1 \otimes \varphi_{\mathfrak{M}})$ is killed by $E(u)^r$. Suppose that $\mathfrak{M}[\frac{1}{u}] := \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$ with Frobenius induced from $\varphi_{\mathfrak{M}}$ is an étale $(\varphi, \mathcal{O}_{\mathcal{E}})$ -module which is free over $\mathcal{O}_{\mathcal{E}}/p^n$. Write $\mathfrak{M}[\frac{1}{t}] := \mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}_n[\frac{1}{t}]$ (note that \mathfrak{M} is t -torsion free). We define the saturation of \mathfrak{M} to be*

$$\text{Sat}(\mathfrak{M}) := \mathfrak{M}[\frac{1}{u}] \cap \mathfrak{M}[\frac{1}{t}],$$

by viewing both $\mathfrak{M}[\frac{1}{u}]$ and $\mathfrak{M}[\frac{1}{t}]$ as \mathfrak{S} -submodules of $\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}_n[\frac{1}{ut}]$. Then, we have $\text{Sat}(\mathfrak{M}) \in \text{Mod}_{\mathfrak{S}}^{\text{tor}, r}(\varphi)$ and $\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} = \text{Sat}(\mathfrak{M}) \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$.

Proof. First, we show $\text{proj.dim}_{\mathfrak{S}}(\text{Sat}(\mathfrak{M})) \leq 1$. By the Auslander-Buchbaum Theorem, it suffices to show that $\text{depth}_{\mathfrak{S}}(\text{Sat}(\mathfrak{M})) \geq 2$. If we have $x, y \in \text{Sat}(\mathfrak{M})$ such that $ux = ty$, then

$$\frac{y}{u} = \frac{x}{t} \in \mathfrak{M}[\frac{1}{u}] \cap \mathfrak{M}[\frac{1}{t}] = \text{Sat}(\mathfrak{M}).$$

Thus, the sequence (u, t) is a regular sequence, and $\text{depth}_{\mathfrak{S}}(\text{Sat}(\mathfrak{M})) \geq 2$.

Since $\varphi : \mathfrak{S} \rightarrow \mathfrak{S}$ is flat, we have

$$\varphi^*(\text{Sat}(\mathfrak{M})) = \mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \text{Sat}(\mathfrak{M}) = \varphi^* \mathfrak{M}[\frac{1}{u}] \cap \varphi^* \mathfrak{M}[\frac{1}{t}].$$

Note that $1 \otimes \varphi_{\mathfrak{M}} : \varphi^* \mathfrak{M} \rightarrow \mathfrak{M}$ is injective since \mathfrak{M} is u -torsion free and $\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$ is an étale $(\varphi, \mathcal{O}_{\mathcal{E}})$ -module. Its cokernel is killed by $E(u)^r$, so there exists a \mathfrak{S} -linear map

$\psi_{\mathfrak{M}} : \mathfrak{M} \rightarrow \varphi^* \mathfrak{M}$ such that $(1 \otimes \varphi_{\mathfrak{M}}) \circ \psi_{\mathfrak{M}} = c_0^{-r} E(u)^r \text{Id}_{\mathfrak{M}}$. This extends naturally to $\psi_{\mathfrak{M}} : \text{Sat}(\mathfrak{M}) \rightarrow \varphi^*(\text{Sat}(\mathfrak{M}))$, and we have $(1 \otimes \varphi_{\mathfrak{M}}) \circ \psi_{\mathfrak{M}} = c_0^{-r} E(u)^r \text{Id}_{\text{Sat}(\mathfrak{M})}$. Thus, $\text{coker}(1 \otimes \varphi_{\mathfrak{M}} : \varphi^*(\text{Sat}(\mathfrak{M})) \rightarrow \text{Sat}(\mathfrak{M}))$ is killed by $E(u)^r$, and $\text{Sat}(\mathfrak{M}) \in \text{Mod}_{\mathfrak{S}}^{\text{tor}, r}(\varphi)$. The identity $\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} = \text{Sat}(\mathfrak{M}) \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$ is clear. \square

Note that when \mathfrak{M} as in Lemma 4.32 is killed by p , then $\text{Sat}(\mathfrak{M})$ is free over $\mathfrak{S}_1 = k[[t, u]]$. Furthermore, if $\mathfrak{M} \in \text{Mod}_{\mathfrak{S}}^{\text{tor}, r}(\varphi)$, then since \mathfrak{M} is p -power torsion and $\text{depth}_{\mathfrak{S}}(\mathfrak{M}) \geq 2$ by the Auslander-Buchbaum Theorem, we have $\mathfrak{M} = \text{Sat}(\mathfrak{M})$.

Lemma 4.33. *For proving Theorem 4.31, it suffices to show the statement for the following case: $f : T_{\mathfrak{S}}^{\vee}(\mathfrak{M}') \rightarrow T_{\mathfrak{S}}^{\vee}(\mathfrak{M})$ is an isomorphism, \mathfrak{M}' is a finite free \mathfrak{S}_n -module, and there exists a morphism of torsion Kisin modules $\mathfrak{g} : \mathfrak{M}' \rightarrow \mathfrak{M}$ such that $T_{\mathfrak{S}}^{\vee}(\mathfrak{g}) = f^{-1}$.*

Proof. Let $M = \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$, $M' = \mathfrak{M}' \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$, and let $\tilde{f} : M \rightarrow M'$ be the morphism of étale $(\varphi, \mathcal{O}_{\mathcal{E}})$ -modules corresponding to f by Proposition 4.20 and Lemma 4.26. We need to show the existence of a constant \mathfrak{c} such that $p^{\mathfrak{c}} \tilde{f}(\mathfrak{M}) \subset \mathfrak{M}'$.

First, by Lemma 4.25, we have a surjection $q : \mathfrak{N} \rightarrow \mathfrak{M}$ in $\text{Mod}_{\mathfrak{S}}^{\text{tor}, r}(\varphi)$ with \mathfrak{N} a finite free \mathfrak{S}_n -module. Let $N = \mathfrak{N} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$ and $\tilde{q} = q \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$. Then $p^{\mathfrak{c}} \tilde{f}(\mathfrak{M}) \subset \mathfrak{M}'$ if and only if $p^{\mathfrak{c}} \tilde{f} \circ \tilde{q}(\mathfrak{N}) \subset \mathfrak{M}'$. Thus, it suffices to prove the theorem when \mathfrak{M} is finite free over \mathfrak{S}_n . Moreover, by taking duality, we see that it suffices to consider the case when \mathfrak{M}' is finite free over \mathfrak{S}_n .

Let Γ be the image of $1 \times (-\tilde{f})$ in $M \times M'$. We have the following exact sequence of étale $(\varphi, \mathcal{O}_{\mathcal{E}})$ -modules:

$$0 \rightarrow \Gamma \rightarrow M \times M' \xrightarrow{q_1} M' \rightarrow 0.$$

Here, q_1 is the map induced by the natural isomorphism $M \xrightarrow{\cong} \Gamma$. Let $\mathfrak{N} = \text{Sat}(q_1(\mathfrak{M} \times \mathfrak{M}'))$,

and let $h : \mathfrak{M} \times \mathfrak{M}' \rightarrow \mathfrak{N}$ be the composite of morphisms

$$\mathfrak{M} \times \mathfrak{M}' \xrightarrow{q_1} q_1(\mathfrak{M} \times \mathfrak{M}') \hookrightarrow \mathfrak{N}$$

of φ -modules. Let $i_1 : \mathfrak{M} \hookrightarrow \mathfrak{M} \times \mathfrak{M}'$ and $i_2 : \mathfrak{M}' \hookrightarrow \mathfrak{M} \times \mathfrak{M}'$ be the natural injections. Note that by Lemma 4.32, $\mathfrak{N} \in \text{Mod}_{\mathfrak{S}}^{\text{tor}, r}(\varphi)$ and $(h \circ i_2) \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} : \mathfrak{M}' \rightarrow \mathfrak{N} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$ is an isomorphism. Moreover, $((h \circ i_2) \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}})^{-1} \circ ((h \circ i_1) \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}) = \tilde{f}$.

Thus, if we can prove Theorem 4.31 for this case so that there exists $\mathfrak{g}' : \mathfrak{N} \rightarrow \mathfrak{M}'$ with $\mathfrak{g}' \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} = p^e((h \circ i_2) \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}})^{-1}$, then we have $\mathfrak{g} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} = p^e \tilde{f}$ for $\mathfrak{g} := \mathfrak{g}' \circ (h \circ i_1)$. \square

Now we consider the case as in Lemma 4.33. The given map $\mathfrak{g} : \mathfrak{M}' \rightarrow \mathfrak{M}$ is injective since $\mathfrak{g} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$ is an isomorphism. We consider \mathfrak{M}' as a submodule of \mathfrak{M} . Then it suffices to prove the following.

Lemma 4.34. *Let $\mathfrak{M}, \mathfrak{M}' \in \text{Mod}_{\mathfrak{S}}^{\text{tor}, r}(\varphi)$ with \mathfrak{M}' finite free over \mathfrak{S}_n such that $\mathfrak{M}' \subset \mathfrak{M}$ and $\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} = \mathfrak{M}' \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$. Then there exists a constant \mathfrak{c} depending only on e and r such that $p^{\mathfrak{c}} \mathfrak{M} \subset \mathfrak{M}'$.*

Proof. By Proposition 4.29 and 4.30, we have the commutative diagram:

$$\begin{array}{ccc} \mathfrak{M}' & \xrightarrow{i} & (T_{\mathfrak{S}}(\mathfrak{M}') \otimes_{\mathbb{Z}_p} \mathfrak{S}_n^{f(r)})^{\mathcal{G}_{\tilde{R}_{\infty}}} \xrightarrow{i^{\vee}} (\mathfrak{M}' \otimes_{\mathfrak{S}_n} \mathfrak{S}_n^{f(2r)}(-r))^{\mathcal{G}_{\tilde{R}_{\infty}}} \\ \downarrow & & \parallel \\ \mathfrak{M} & \xrightarrow{i_{\mathfrak{M}}} & (T_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathfrak{S}_n^{f(r)})^{\mathcal{G}_{\tilde{R}_{\infty}}} \end{array}$$

Since \mathfrak{M}' is finite free over \mathfrak{S}_n , $(\mathfrak{M}' \otimes_{\mathfrak{S}_n} \mathfrak{S}_n^{f(2r)}(-r))^{\mathcal{G}_{\tilde{R}_{\infty}}} = \mathfrak{M}' \otimes_{\mathfrak{S}_n} (\mathfrak{S}_n^{f(2r)}(-r))^{\mathcal{G}_{\tilde{R}_{\infty}}}$. Thus, by Proposition 4.30, we have

$$\hat{i}^{\vee} \circ \hat{i}(\mathfrak{M}') = \mathfrak{M}' \otimes_{\mathfrak{S}_n} \mathfrak{S}_n \cdot \mathfrak{t}^r \subset \hat{i}^{\vee} \circ \hat{i}_{\mathfrak{M}}(\mathfrak{M}) \subset \mathfrak{M}' \otimes_{\mathfrak{S}_n} (\mathfrak{S}_n^{f(2r)}(-r))^{\mathcal{G}_{\tilde{R}_{\infty}}},$$

so it suffices to prove $p^c(\mathfrak{S}_n^{f(2r)}(-r))^{\mathcal{G}_{\tilde{R}_\infty}} \subset \mathfrak{S}_n \cdot \mathfrak{t}^r$. Note further that we have

$(\mathcal{O}_{\mathcal{E},n}^{\text{ur}}(-r))^{\mathcal{G}_{\tilde{R}_\infty}} = \mathcal{O}_{\mathcal{E},n} \cdot \mathfrak{t}^r$ from the isomorphism $\mathfrak{t}^r \otimes_{\mathfrak{S}^{\text{ur}}} \mathcal{O}_{\mathcal{E}}^{\text{ur}} : \mathcal{O}_{\mathcal{E},n}^{\text{ur},\vee} \xrightarrow{\cong} \mathcal{O}_{\mathcal{E},n}^{\text{ur}}(-r)$ and Lemma 4.18. Since $(\mathfrak{S}_n^{f(2r)}(-r))^{\mathcal{G}_{\tilde{R}_\infty}} \subset (\mathcal{O}_{\mathcal{E},n}^{\text{ur}}(-r))^{\mathcal{G}_{\tilde{R}_\infty}}$, it suffices to show that there exists a constant \mathfrak{c} depending only on e and r such that

$$p^c(\mathfrak{S}_n^{f(2r)} \cap \mathcal{O}_{\mathcal{E},n} \cdot \mathfrak{t}^r) \subset \mathfrak{S}_n \cdot \mathfrak{t}^r.$$

We let $\mathfrak{c} = 0$ if $er < p - 1$ and $\mathfrak{c} = \lfloor \frac{er}{p-1} \rfloor (2^{2r}(r \lfloor \frac{er}{p-1} \rfloor + 1) - 1) + 1$ otherwise. Consider the base change $b_g : R_0 \rightarrow R'_0$ as above. Let $F_{\mathfrak{S}'}^{f(r)}(\mathfrak{S}_n^{\text{ur}})$ be the set of φ -stable finite \mathfrak{S}'_n -submodules of $\mathfrak{S}_n^{\text{ur}}$ such that $\text{coker}(1 \otimes \varphi)$ is killed by $E(u)^r$. Let $\mathfrak{S}_n'^{f(r)} := \bigcup_{\mathfrak{N} \in F_{\mathfrak{S}'}^{f(r)}(\mathfrak{S}_n^{\text{ur}})} \mathfrak{N}$.

Let $x \in \mathfrak{S}_n^{f(2r)} \cap \mathcal{O}_{\mathcal{E},n} \cdot \mathfrak{t}^r$, and let $\mathfrak{N} \in F_{\mathfrak{S}}^{f(2r)}(\mathfrak{S}_n^{\text{ur}})$ containing x . Then $\mathfrak{N} \otimes_{\mathfrak{S}, b_g} \mathfrak{S}' \in F_{\mathfrak{S}'}^{f(2r)}(\mathfrak{S}_n^{\text{ur}})$, so $x \in \mathfrak{S}_n'^{f(2r)} \cap \mathcal{O}_{\mathcal{E}',n} \cdot \mathfrak{t}^r$. Since $R'_0 \cong W(k((t))^{\text{perf}})$, we see from the proof of [25, Theorem 2.4.2] that $p^c x \in \mathfrak{S}'_n \cdot \mathfrak{t}^r$. Thus,

$$p^c x \in (\mathcal{O}_{\mathcal{E},n} \cdot \mathfrak{t}^r) \cap (\mathfrak{S}'_n \cdot \mathfrak{t}^r) = (\mathcal{O}_{\mathcal{E},n} \cap \mathfrak{S}'_n) \cdot \mathfrak{t}^r = \mathfrak{S}_n \cdot \mathfrak{t}^r.$$

□

The following corollary is immediate.

Corollary 4.35. *Suppose that $\mathfrak{M}, \tilde{\mathfrak{M}} \in \text{Mod}_{\mathfrak{S}}^{\text{tor}, r}(\varphi)$ such that $T_{\mathfrak{S}}^{\vee}(\mathfrak{M}) \cong T_{\mathfrak{S}}^{\vee}(\tilde{\mathfrak{M}})$. If we identify $\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} = \tilde{\mathfrak{M}} \otimes \mathcal{O}_{\mathcal{E}}$ (by Proposition 4.20 and Lemma 4.26), then $p^c \tilde{\mathfrak{M}} \subset \mathfrak{M}$ and $p^c \mathfrak{M} \subset \tilde{\mathfrak{M}}$.*

As another consequence of Theorem 4.31, we have the following generalization of [18, Proposition 2.1.12] to the case $R = \mathcal{O}_K[[t]]$.

Corollary 4.36. *The functor $T_{\mathfrak{S}}^{\vee} : \text{Mod}_{\mathfrak{S}}^r(\varphi) \rightarrow \text{Rep}_{\mathbb{Z}_p}^{\text{free}}(\mathcal{G}_{\tilde{R}_\infty})$ is fully faithful.*

Proof. Let $\mathfrak{M}, \mathfrak{N} \in \text{Mod}_{\mathfrak{S}}^r(\varphi)$, and let $\tilde{f} : T_{\mathfrak{S}}^{\vee}(\mathfrak{N}) \rightarrow T_{\mathfrak{S}}^{\vee}(\mathfrak{M})$ be a morphism of $\mathbb{Z}_p[\mathcal{G}_{\tilde{R}_{\infty}}]$ -modules. Then by Proposition 4.20 and Lemma 4.26, we have a morphism $f : \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} \rightarrow \mathfrak{N} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$ of étale $(\varphi, \mathcal{O}_{\mathcal{E}})$ -modules such that $T^{\vee}(f) = \tilde{f}$. We need to show that $f(\mathfrak{M}) \subset \mathfrak{N}$. By Theorem 4.31, we have $p^{\mathfrak{c}} f(\mathfrak{M}/p^n) \subset \mathfrak{N}/p^n$ for all $n \geq \mathfrak{c}$. Since $\mathfrak{M}, \mathfrak{N}$ are finite free over \mathfrak{S} , we then have $f(\mathfrak{M}/p^{n-\mathfrak{c}}) \subset \mathfrak{N}/p^{n-\mathfrak{c}}$ for all $n \geq \mathfrak{c}$. Thus, $f(\mathfrak{M}) \subset \mathfrak{N}$. \square

4.3.2 Proof of Theorem 4.4 and 4.5 for $R = \mathcal{O}_K[[t]]$

With the weak full faithfulness of the functor $T_{\mathfrak{S}}^{\vee}$, we can now prove Theorem 4.4. Let $\mathfrak{m} \subset R = \mathcal{O}_K[[t]]$ be the maximal ideal, and let $U := \text{Spec} R \setminus \{\mathfrak{m}\}$. We begin by proving the following lemma about general properties of finite locally free group schemes over R , which is used in [37] to prove purity/non-purity results for p -divisible groups. Let $b_g : R_0 \rightarrow R'_0 \cong W(k((t))^{\text{perf}})$ be the base change as given in Section 4.3.1.

Lemma 4.37. *Let $\beta : H_1 \rightarrow H_2$ be a morphism of finite locally free group schemes over R , and let $\mathfrak{M}_i = \mathfrak{M}^*(H_i)$ for $i = 1, 2$. Then, $\beta|_U : H_{1,U} \rightarrow H_{2,U}$ is a monomorphism (resp. an isomorphism) if and only if*

$$\mathfrak{M}^*(\beta) \otimes_{\mathfrak{S}, b_g} W(k((t))^{\text{perf}})[[u]] : \mathfrak{M}_2 \otimes_{\mathfrak{S}, b_g} W(k((t))^{\text{perf}})[[u]] \rightarrow \mathfrak{M}_1 \otimes_{\mathfrak{S}, b_g} W(k((t))^{\text{perf}})[[u]]$$

induced by b_g is an epimorphism (resp. isomorphism).

Proof. First, we claim that $\beta|_U$ is an epimorphism (resp. a monomorphism) if and only if

$$\beta \times_{R, b_g} R' : H_1 \times_{R, b_g} R' \rightarrow H_2 \times_{R, b_g} R'$$

is an epimorphism (resp. a monomorphism). By Cartier duality, it suffices to show the claim for epimorphisms (i.e. faithfully flat maps). Suppose that $\beta \times_{R, b_g} R'$ is faithfully flat.

Note that the natural map $R_{(\varpi)} \rightarrow R'$ is flat by the fiber criterion of flatness, and thus it is faithfully flat since it is local. Therefore, the map

$$\beta \times_R R_{(\varpi)} : H_1 \times_R R_{(\varpi)} \rightarrow H_2 \times_R R_{(\varpi)}$$

is faithfully flat. In particular, the map corresponding to β between finite free R -algebras locally defining H_i is injective, since its localization at (ϖ) is injective. Let $\mathfrak{q} \in U$ be a prime ideal not containing ϖ , and let $\hat{R}_{\mathfrak{q}}$ be the \mathfrak{q} -adic completion of $R_{\mathfrak{q}}$. Since $H_1 \times_R \hat{R}_{\mathfrak{q}}$ and $H_2 \times_R \hat{R}_{\mathfrak{q}}$ are étale over $\hat{R}_{\mathfrak{q}}$, the map $\beta \times_R \hat{R}_{\mathfrak{q}}$ is étale. Since the corresponding ring morphisms are injective and finite, $\beta \times_R \hat{R}_{\mathfrak{q}}$ is faithfully flat, and so $\beta \times_R R_{\mathfrak{q}}$ is faithfully flat. Thus, $\beta|_U$ is faithfully flat. The converse is clear.

Now, note that $\mathfrak{M}^*(H_i \times_{R, b_g} R') = \mathfrak{M}_i \otimes_{\mathfrak{S}, b_g} W(k((t))^{\text{perf}})[[u]]$ as torsion Kisin modules over $\mathfrak{S}' = W(k((t))^{\text{perf}})[[u]]$ by Proposition 4.13. And if $\mathfrak{M}^*(\beta) \otimes_{\mathfrak{S}, b_g} W(k((t))^{\text{perf}})[[u]]$ is an epimorphism, then its kernel has \mathfrak{S}' -projective dimension ≤ 1 , so it is a torsion Kisin module of height 1 over \mathfrak{S}' . Hence, the result follows. \square

Lemma 4.38. *The category of finite locally free group schemes over U is equivalent to the category of finite locally free group schemes over R .*

Proof. This follows directly from [11, Lemma 6.2]. \square

Lemma 4.39. *Let H_2 be a finite locally free group scheme over R . Suppose we have a monomorphism $\beta_{R[\frac{1}{p}]} : H_{1, R[\frac{1}{p}]} \rightarrow H_{2, R[\frac{1}{p}]}$ of finite locally free group schemes over $R[\frac{1}{p}]$. Let $\mathfrak{M}_2 = \mathfrak{M}^*(H_2)$ be the torsion Kisin module for H_2 , and let $\alpha : \mathfrak{M}_2 \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} \rightarrow M_1$ be the morphism of étale $(\varphi, \mathcal{O}_{\mathcal{E}})$ -modules corresponding to $\beta_{R[\frac{1}{p}]}$ via Proposition 4.20 (so that α is surjective). We further assume that M_1 is free over $\mathcal{O}_{\mathcal{E}}/p^n$ for some n . Then, the finite locally free group scheme corresponding to $\text{Sat}(\alpha(\mathfrak{M}_2))$ agrees with $H_{1, R[\frac{1}{p}]}$ over $R[\frac{1}{p}]$.*

Proof. Let $H'_{1,U}$ be the scheme theoretic closure of $H_{1,R[\frac{1}{p}]} \times_{\text{Spec} R} U$ induced by $\beta_{R[\frac{1}{p}]} \times_{\text{Spec} R} U$, as in [32, Section 2.1]. By the discrete valuation ring cases proved in [32, Section 2.1], $H'_{1,U}$ is a finite locally free group scheme over U , which extends to a finite locally free group scheme H'_1 over R by Lemma 4.38. The corresponding Hopf algebra structure for H'_1 can be described as follows.

Let H_2 be locally defined by a R -algebra A_2 , and let $H_{1,R[\frac{1}{p}]}$ be defined by $R[\frac{1}{p}]$ -algebra $A_2[\frac{1}{p}]/I$ via $\beta_{R[\frac{1}{p}]}$ for an ideal I of $A_2[\frac{1}{p}]$. Let J be the pre-image of I in A_2 . Then, by the proof of [11, Lemma 6.2], the R -algebra $(A_2/J)[\frac{1}{p}] \cap (A_2/J)[\frac{1}{t}]$ has the induced Hopf algebra structure defining the group scheme H'_1 over R . Note that $(A_2/J)[\frac{1}{p}] \cap (A_2/J)[\frac{1}{t}]$ has R -depth 2 and therefore is projective over R by the Auslander-Buchbaum Theorem. We have $H'_{1,R[\frac{1}{p}]} = H_{1,R[\frac{1}{p}]}$.

Let $\mathfrak{M}_1 := \mathfrak{M}^*(H'_1)$ be the corresponding torsion Kisin module. Then, $\mathfrak{M}_1[\frac{1}{u}] = \alpha(\mathfrak{M}_2)[\frac{1}{u}]$. Furthermore, $\mathfrak{M}_1 \otimes_{\mathfrak{S}, b_g} W(k((t))^{\text{perf}})[[u]] = \alpha(\mathfrak{M}_2) \otimes_{\mathfrak{S}, b_g} W(k((t))^{\text{perf}})[[u]]$ by the construction. Note that $\{\text{Spec}(W(k((t))^{\text{perf}})[[u]]/p^n), \text{Spec}(\mathcal{O}_{\mathcal{E},n}[\frac{1}{t}])\}$ is an fpqc covering of $\text{Spec}(\mathfrak{S}_n[\frac{1}{t}])$. So by fpqc descent, we have $\mathfrak{M}_1[\frac{1}{t}] = \alpha(\mathfrak{M}_2)[\frac{1}{t}]$. Thus, $\mathfrak{M}_1 = \text{Sat}(\alpha(\mathfrak{M}_2))$, and the result follows. \square

Let $\mathfrak{M} \in \text{Mod}_{\mathfrak{S}}^{\text{tor},r}(\varphi)$ such that $M := \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$ is a finite free $\mathcal{O}_{\mathcal{E},n}$ -module. For $0 \leq i \leq j \leq n$, we let

$$\mathfrak{M}^{i,j} := \text{Sat}(\ker(p^i \mathfrak{M} \xrightarrow{p^{j-i}} p^j \mathfrak{M})).$$

By Lemma 4.32, $\mathfrak{M}^{i,j} \in \text{Mod}_{\mathfrak{S}}^{\text{tor},r}(\varphi)$ and $\mathfrak{M}^{i,j} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} \cong M/p^{j-i}$. Note that for any $s \leq j-i$, we have $p^s \mathfrak{M}^{i,j}[\frac{1}{t}] = \mathfrak{M}^{s+i,j}[\frac{1}{t}]$. Furthermore, for any $l \geq 0$ such that $l+j \leq n$, the natural injections $p^{i+l} \mathfrak{M} \hookrightarrow p^i \mathfrak{M}$ and $p^{j+l} \mathfrak{M} \hookrightarrow p^j \mathfrak{M}$ induce the map

$$\alpha^{i,j,l} : \mathfrak{M}^{i+l,j+1} \rightarrow \mathfrak{M}^{i,j}.$$

$\alpha^{i,j,l} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$ is an isomorphism. In particular, for $l = 1$ and $i = j$, we have the following chain

$$\mathfrak{M}^{n-1,n} \subset \dots \subset \mathfrak{M}^{1,2} \subset \mathfrak{M}^{0,1} \subset M/p \quad (4.2)$$

such that $\mathfrak{M}^{i,i+1} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} = M/p$ for $0 \leq i \leq n-1$.

Lemma 4.40. (cf. [25, Lemma 4.2.4]) *In the chain (4.2) as above, if there exist i_0 and s such that*

$$\mathfrak{M}^{i_0+s-1,i_0+s} = \dots = \mathfrak{M}^{i_0+1,i_0+2} = \mathfrak{M}^{i_0,i_0+1},$$

then $\mathfrak{M}^{i_0,i_0+s}[\frac{1}{t}]$ is finite free over $\mathfrak{S}_s[\frac{1}{t}]$.

Proof. This can be proved by essentially the same argument as in the proof of [25, Lemma 4.2.4], as follows.

For any $0 \leq m \leq s$, let $\Gamma_m = \mathfrak{M}^{i_0+(s-m),i_0+s}[\frac{1}{t}]$. We have $\Gamma_m = p^{s-m}\Gamma_s$. We claim that $\Gamma_{m+1}/p^m = \Gamma_m$. For this, consider the following commutative diagram:

$$\begin{array}{ccc} \Gamma_{m+1} & \xrightarrow{p^m} & \Gamma_{m+1} \\ \downarrow \beta & & \uparrow \gamma \\ \Gamma_1 & \xrightarrow{\alpha} & \mathfrak{M}^{i_0+(s-m-1),i_0+(s-m)}[\frac{1}{t}] \end{array}$$

Here, $\alpha := \alpha^{i_0+(s-m-1),i_0+(s-m),m} \otimes_{\mathfrak{S}} \mathfrak{S}[\frac{1}{t}]$, which is an isomorphism by the assumption. β is the map induced by $p^m : p^{i_0+(s-m-1)}\mathfrak{M} \rightarrow p^{i_0+(s-1)}\mathfrak{M}$ and it is surjective. γ is the map induced by $p^m : p^{i_0+(s-m)}\mathfrak{M} \rightarrow p^{i_0+s}\mathfrak{M}$ and it is injective. The diagram is commutative since it is commutative after tensoring with $\mathcal{O}_{\mathcal{E}}$. Since β is surjective and α is an isomorphism, $\Gamma_{m+1}/p^m = \text{coker}(\gamma)$. On the other hand, by the Snake Lemma and diagram chasing, we have $\text{coker}(\gamma) = \ker(p^{i_0+(s-m)}\mathfrak{M} \xrightarrow{p^m} p^{i_0+s}\mathfrak{M})[\frac{1}{t}] = \Gamma_m$. Thus, $\Gamma_{m+1}/p^m = \Gamma_m$.

Now, we prove that Γ_m is finite free over $\mathfrak{S}_m[\frac{1}{t}]$ by induction on m . Since Γ_1 is finite

torsion free as a $k[[u, t]][\frac{1}{t}]$ -module, the case $m = 1$ is obvious. Suppose that Γ_m is a free $\mathfrak{S}_m[\frac{1}{t}]$ -module of rank h . Choose $x_1, \dots, x_h \in \Gamma_{m+1}$ such that $\{px_1, \dots, px_h\}$ is a basis of Γ_m . By Nakayama's lemma, x_1, \dots, x_h generate Γ_{m+1} . Thus, we have a natural surjection $f : \bigoplus_{i=1}^h \mathfrak{S}_{m+1}[\frac{1}{t}] \rightarrow \Gamma_{m+1}$. Since M/p^{m+1} is a free $\mathcal{O}_{\mathcal{E}, m+1}$ -module of rank h , $f \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$ is a bijection. Since Γ_{m+1} is u -torsion free, f is injective, and therefore Γ_{m+1} is finite free over $\mathfrak{S}_{m+1}[\frac{1}{t}]$. Thus, $\mathfrak{M}^{i_0, i_0+s}[\frac{1}{t}] = \Gamma_s$ is finite free over $\mathfrak{S}_s[\frac{1}{t}]$. \square

Lemma 4.41. *Let $\mathfrak{M} \in \text{Mod}_{\mathfrak{S}}^{\text{tor}, r}(\varphi)$ such that $M := \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$ is a finite free $\mathcal{O}_{\mathcal{E}, n}$ -module. Let $n \geq 2\mathfrak{c} + 1$. Then $\tilde{\mathfrak{M}} := \text{Sat}(\ker(p^{\mathfrak{c}}\mathfrak{M} \xrightarrow{p^{n-2\mathfrak{c}}} p^{n-\mathfrak{c}}\mathfrak{M})) \in \text{Mod}_{\mathfrak{S}}^{\text{tor}, r}(\varphi)$ is a $\mathfrak{S}_{n-2\mathfrak{c}}$ -module such that $\tilde{\mathfrak{M}} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} \cong M/p^{n-2\mathfrak{c}}$ and $\tilde{\mathfrak{M}}[\frac{1}{t}]$ is finite free over $\mathfrak{S}_{n-2\mathfrak{c}}[\frac{1}{t}]$.*

Proof. We use the same notation as above. By Lemma 4.40, it suffices to show that $\mathfrak{M}^{n-\mathfrak{c}-1, n-\mathfrak{c}} = \mathfrak{M}^{\mathfrak{c}, \mathfrak{c}+1}$. Note that we have a natural injection $p^{n-\mathfrak{c}-1}\mathfrak{M} \hookrightarrow \mathfrak{M}^{0, \mathfrak{c}+1}$, which induces the injection $f : \text{Sat}(p^{n-\mathfrak{c}-1}\mathfrak{M}) \hookrightarrow \mathfrak{M}^{0, \mathfrak{c}+1}$. Since $f \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$ is an isomorphism, we have $p^{\mathfrak{c}}\mathfrak{M}^{0, \mathfrak{c}+1} \subset \text{Sat}(p^{n-\mathfrak{c}-1}\mathfrak{M})$ by Corollary 4.35. But $p^{\mathfrak{c}}\mathfrak{M}^{0, \mathfrak{c}+1}$ is killed by p , so $\mathfrak{M}^{\mathfrak{c}, \mathfrak{c}+1} = \text{Sat}(p^{\mathfrak{c}}\mathfrak{M}^{0, \mathfrak{c}+1}) \subset \mathfrak{M}^{n-\mathfrak{c}-1, n-\mathfrak{c}}$. \square

Let $G = (G_n)_{n \geq 0}$ be a p -divisible group over $R[\frac{1}{p}]$, and suppose that for each n , G_n extends to a finite locally free group scheme $G_{n, R}$ over R . Let $M \in \text{Mod}_{\mathcal{O}_{\mathcal{E}}}^{\text{ét}, \text{pr}}(\varphi)$ be the étale $(\varphi, \mathcal{O}_{\mathcal{E}})$ -module associated with G via the equivalence in Proposition 4.20, and let $M_n := M/p^n \in \text{Mod}_{\mathcal{O}_{\mathcal{E}}}^{\text{ét}, \text{tor}}(\varphi)$. For each n , let $\mathfrak{M}_n := \mathfrak{M}^*(G_{n, R}) \in \text{Mod}_{\mathfrak{S}}^{\text{tor}, 1}(\varphi)$ be the torsion Kisin module of height 1. We have $\mathfrak{M}_n \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} \cong M_n$.

Proof of Theorem 4.4. Let $\mathfrak{M}'_n := \mathfrak{M}_{n+3\mathfrak{c}}^{2\mathfrak{c}, n+2\mathfrak{c}} \in \text{Mod}_{\mathfrak{S}}^{\text{tor}, 1}(\varphi)$. Note that $\mathfrak{M}'_n \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} \cong M_n$. By Lemma 4.41, $\mathfrak{M}_{n+3\mathfrak{c}}^{\mathfrak{c}, n+2\mathfrak{c}}[\frac{1}{t}]$ is finite free over $\mathfrak{S}_{n+\mathfrak{c}}[\frac{1}{t}]$, so $\mathfrak{M}'_n[\frac{1}{t}] = p^{\mathfrak{c}}\mathfrak{M}_{n+3\mathfrak{c}}^{\mathfrak{c}, n+2\mathfrak{c}}[\frac{1}{t}]$ is finite free over $\mathfrak{S}_n[\frac{1}{t}]$. Since $\mathfrak{M}_{n+1+3\mathfrak{c}}^{\mathfrak{c}+1, n+1+2\mathfrak{c}} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} = \mathfrak{M}_{n+3\mathfrak{c}}^{\mathfrak{c}, n+2\mathfrak{c}} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$, we have $p^{\mathfrak{c}}\mathfrak{M}_{n+1+3\mathfrak{c}}^{\mathfrak{c}+1, n+1+2\mathfrak{c}} \subset \mathfrak{M}_{n+3\mathfrak{c}}^{\mathfrak{c}, n+2\mathfrak{c}}$ and $p^{\mathfrak{c}}\mathfrak{M}_{n+3\mathfrak{c}}^{\mathfrak{c}, n+2\mathfrak{c}} \subset \mathfrak{M}_{n+1+3\mathfrak{c}}^{\mathfrak{c}+1, n+1+2\mathfrak{c}}$ by Corollary 4.35. Since both $\mathfrak{M}_{n+1+3\mathfrak{c}}^{\mathfrak{c}+1, n+1+2\mathfrak{c}}[\frac{1}{t}]$ and $\mathfrak{M}_{n+3\mathfrak{c}}^{\mathfrak{c}, n+2\mathfrak{c}}[\frac{1}{t}]$

are finite free $\mathfrak{S}_{n+\mathfrak{c}}[\frac{1}{t}]$ -modules, we have $p^{\mathfrak{c}}\mathfrak{M}_{n+1+3\mathfrak{c}}^{\mathfrak{c}+1,n+1+2\mathfrak{c}}[\frac{1}{t}] \cong p^{\mathfrak{c}}\mathfrak{M}_{n+3\mathfrak{c}}^{\mathfrak{c},n+2\mathfrak{c}}[\frac{1}{t}]$. Thus,

$$p\mathfrak{M}'_{n+1}[\frac{1}{t}] = p^{\mathfrak{c}}p\mathfrak{M}_{n+1+3\mathfrak{c}}^{\mathfrak{c},n+1+2\mathfrak{c}}[\frac{1}{t}] = p^{\mathfrak{c}}\mathfrak{M}_{n+1+3\mathfrak{c}}^{\mathfrak{c}+1,n+1+2\mathfrak{c}}[\frac{1}{t}] \cong p^{\mathfrak{c}}\mathfrak{M}_{n+3\mathfrak{c}}^{\mathfrak{c},n+2\mathfrak{c}}[\frac{1}{t}] = \mathfrak{M}'_n[\frac{1}{t}],$$

i.e., $p\mathfrak{M}'_{n+1}[\frac{1}{t}] \cong \mathfrak{M}'_n[\frac{1}{t}]$. This implies

$$p\mathfrak{M}'_{n+1} \otimes_{\mathfrak{S}, b_g} \mathfrak{S}' \cong \mathfrak{M}'_n \otimes_{\mathfrak{S}, b_g} \mathfrak{S}'. \quad (4.3)$$

Let $G_{U,n}$ be the restriction to U of the finite locally free group scheme over R corresponding to \mathfrak{M}'_n . Note that by Lemma 4.38, any morphism of finite locally free group schemes over U extends uniquely to a morphism of finite locally free group scheme over R . Since $\mathfrak{M}'_n \otimes_{\mathfrak{S}, b_g} \mathfrak{S}'$ is finite free over \mathfrak{S}'_n , we then have by Lemma 4.37 the following exact sequence of finite locally free group schemes over U for each $1 \leq i \leq n-1$:

$$0 \rightarrow G_{U,n}[p^i] \rightarrow G_{U,n} \rightarrow G_{U,n}[p^{n-i}] \rightarrow 0.$$

By Corollary 4.35, we have an injective morphism of φ -modules $\text{Sat}(p^{\mathfrak{c}+1}\mathfrak{M}'_{n+1}) \rightarrow \text{Sat}(\mathfrak{M}'_n) = \mathfrak{M}'_n$ since $p\mathfrak{M}'_{n+1} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} \cong \mathfrak{M}'_n \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$, and by (4.3) and Lemma 4.37, this induces the isomorphism $G_{U,n+1}[p^i] \cong G_{U,n}[p^i]$ for each $1 \leq i \leq n-\mathfrak{c}$. We thus obtain a p -divisible group $(G_{U,n+\mathfrak{c}}[p^n])_{n \geq 1}$ over U , and we see from the construction of \mathfrak{M}'_n and Lemma 4.39 that it is an extension of G .

It remains to show the uniqueness. Suppose $(G''_{U,n})_{n \geq 1}$ is another p -divisible group over U extending G . Let G''_n be the finite locally free group scheme over R extending $G''_{U,n}$ by Lemma 4.38, and let $\mathfrak{M}''_n = \mathfrak{M}^*(G''_n)$. We have $\mathfrak{M}'_n \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} = \mathfrak{M}''_n \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$. Since both $\mathfrak{M}'_n \otimes_{\mathfrak{S}, b_g} \mathfrak{S}'$ and $\mathfrak{M}''_n \otimes_{\mathfrak{S}, b_g} \mathfrak{S}'$ are free over \mathfrak{S}'_n , we see by Corollary 4.35 (applied to the index $n+\mathfrak{c}$ for each $n \geq 1$) that $\mathfrak{M}'_n \otimes_{\mathfrak{S}, b_g} \mathfrak{S}' = \mathfrak{M}''_n \otimes_{\mathfrak{S}, b_g} \mathfrak{S}'$. Hence, by the

fpqc descent as in the proof of Lemma 4.39, we have $\mathfrak{M}'_n = \mathfrak{M}''_n$. This implies $G_{U,n+\mathfrak{c}}[p^n] \cong G''_{U,n}$. \square

We now recall some results about purity for p -divisible groups proved in [37].

Definition 4.42. (cf. [37, Definition 2]) Let S be a Noetherian local ring with maximal ideal \mathfrak{n} such that $\text{depth} S \geq 2$. We say that S is *p -quasi-healthy* if each p -divisible group over $\text{Spec} S \setminus \{\mathfrak{n}\}$ extends uniquely to a p -divisible group over $\text{Spec} S$.

Lemma 4.43. (cf. [37, Lemma 20]) *Let S be a complete Noetherian local ring of dimension 2 with maximal ideal \mathfrak{n} . Then S is p -quasi-healthy if and only if each short exact sequence of finite locally free group schemes over $\text{Spec} S \setminus \{\mathfrak{n}\}$ extends uniquely to a short exact sequence of finite locally free group schemes over $\text{Spec} S$.*

Proof. This is a part of [37, Lemma 20]. \square

As a corollary to Lemma 4.43, the following is proved in [37].

Corollary 4.44. (cf. [37, Corollary 4]) *Suppose that the ramification index $e \leq p - 1$. Then, $R = \mathcal{O}_K[[t]]$ is p -quasi-healthy.*

Thus, we see from Theorem 4.4 that in the case $e \leq p - 1$, G extends to a p -divisible group over R uniquely up to isomorphism. This completes the proof for Theorem 4.5 for $R = \mathcal{O}_K[[t]]$.

4.3.3 A Counter-example to Extending G over R

In this section, we show that G as in Theorem 4.4 does not extend to the whole $\text{Spec} R$ in general when $e \geq p$. For this, we use the counter-example to p -quasi-healthiness given in [37], which we recall here.

Proposition 4.45. (cf. [37, Theorem 28]) *Let $R = \mathcal{O}_K[[t]]$ with the ramification index $e \geq p$. Then, R is not p -quasi-healthy.*

Proof. We first construct a homomorphism $\beta : D \rightarrow H$ of finite locally free group schemes over R which is not a monomorphism but whose restriction to U is a monomorphism. Let $\mathfrak{M} = \mathfrak{S}_1^3 \in \text{Mod}_{\mathfrak{S}}^{\text{tor},1}(\varphi)$ with $\varphi_{\mathfrak{M}}$ given by the following matrix:

$$\begin{pmatrix} 0 & 0 & u^p \\ t - t^p u^{p-1} & u & (u - t^{p-1})(t - t^p u^{p-1}) \\ u^{p-1} & 0 & u^{p-1}(u - t^{p-1}) \end{pmatrix}.$$

Then $\text{coker}(1 \otimes \varphi_{\mathfrak{M}})$ is killed by u^p . We let H be the finite locally free group scheme over R corresponding to \mathfrak{M} .

Let $\mathfrak{N} = \mathfrak{S}_1$ with $\varphi_{\mathfrak{N}}$ given by the multiplication by u^p , and let D be the finite locally free group scheme corresponding to \mathfrak{N} . The \mathfrak{S} -linear map $\alpha : \mathfrak{M} \rightarrow \mathfrak{N}$ given by the matrix (t, u, tu) is φ -equivariant. Since α is not surjective, the homomorphism $\beta : D \rightarrow H$ associated to α is not a monomorphism. Note that the natural base change of α by $b_g : \mathfrak{S} \rightarrow W(k((t))^{\text{perf}})[[u]]$ is surjective. This shows by Lemma 4.37 that $\beta_U : D_U \rightarrow H_U$ is a monomorphism.

Now, by [2, Theorem 3.1.1] (or by Lemma 4.25), there exists an embedding $h : H \rightarrow G'_R$ into a p -divisible group G'_R over R , and denote by G'_U the restriction of G'_R to U . Then $h_U \circ \beta_U : D_U \rightarrow G'_U$ is an embedding, and let $G_U = G'_U / D_U$, which is a p -divisible group over U . We claim that G_U does not extend to a p -divisible group over $\text{Spec} R$:

Suppose that G_U extends to G_R over R . Since the quotient map $f_U : G'_U \rightarrow G_U$ is an isogeny, there exists a morphism $g_U : G_U \rightarrow G'_U$ such that $f_U \circ g_U = p^N$ and $g_U \circ f_U = p^N$ for some integer N . Then by Lemma 4.38, the induced morphisms $f : G'_R \rightarrow G_R$ and

$g : G_R \rightarrow G'_R$ satisfy $f \circ g = p^N$ and $g \circ f = p^N$. So f is an isogeny, and $\ker f$ is a finite locally free group scheme. But $\ker f|_U \cong D_U$, so $\ker f \cong D$. In particular, $h \circ \beta : D \rightarrow G'_R$ is a monomorphism, which contradicts that β is not a monomorphism. \square

Remark 4.46. Note that the proof of Proposition 4.45 also shows that over R , the scheme theoretic closure as in [32] does not yield a finite locally free group scheme in general. For D, H as in the proof, the scheme theoretic closure of $\beta_{R[\frac{1}{p}]} : D_{R[\frac{1}{p}]} \rightarrow H_{R[\frac{1}{p}]}$ inside H should be equal to D if it were a finite locally free scheme, since every finite locally free group scheme over U extends uniquely to that over R . However, $\beta : D \rightarrow H$ is not a monomorphism as shown in the proof.

We continue to assume that $e \geq p$. Let G_U be the p -divisible group over U given in Proposition 4.45, so that G_U does not extend to a p -divisible group over R . For each n , the finite locally free group scheme $G_U[p^n]$ over U extends uniquely to $\text{Spec} R$ by Lemma 4.38, and let $\mathfrak{M}_n \in \text{Mod}_{\mathfrak{S}}^{\text{tor}, 1}(\varphi)$ be the corresponding torsion Kisin module. Let G be the restriction of G_U to $\text{Spec} R[\frac{1}{p}]$.

Suppose that G extends to a p -divisible group over R , and let $\mathfrak{M}' \in \text{Mod}_{\mathfrak{S}}^1(\varphi)$ be the corresponding Kisin module. Let $\mathfrak{M}'_n = \mathfrak{M}'/p^n \in \text{Mod}_{\mathfrak{S}}^{\text{tor}, 1}(\varphi)$, which is a finite free \mathfrak{S}_n -module. Note that $\mathfrak{M}_n \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} \cong \mathfrak{M}'_n \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$ as the corresponding finite locally free group schemes agree over the generic fiber $R[\frac{1}{p}]$. Thus, the base change $b_g : \mathfrak{S} \rightarrow \mathfrak{S}'$ induces an isomorphism

$$\mathfrak{M}_n \otimes_{\mathfrak{S}, b_g} \mathfrak{S}' \cong \mathfrak{M}'_n \otimes_{\mathfrak{S}, b_g} \mathfrak{S}' \quad (4.4)$$

as φ -modules by [25, Corollary 4.2.5].

By Corollary 4.35, we have an injection $p^c(\mathfrak{M}'_n)^{\vee} = (\mathfrak{M}'_{n-c})^{\vee} \hookrightarrow \mathfrak{M}_n^{\vee}$. Note that we have the natural map of torsion Kisin modules $\mathfrak{M}_{n-c} \rightarrow \mathfrak{M}_n$ induced from the quotient map $G_U[p^n] \rightarrow G_U[p^{n-c}]$, and let $\mathfrak{M}_n^{\vee} \rightarrow \mathfrak{M}_{n-c}^{\vee}$ be the dual map. Consider the composite map

$(\mathfrak{M}_{n-\mathfrak{c}})^{\vee} \hookrightarrow \mathfrak{M}_n^{\vee} \rightarrow \mathfrak{M}_{n-\mathfrak{c}}^{\vee}$. By taking dual, we get the morphism $\mathfrak{M}_{n-\mathfrak{c}} \rightarrow \mathfrak{M}_{n-\mathfrak{c}}'$. Then, by (4.4) and Lemma 4.37, the corresponding morphism restricted to U is an isomorphism. So $\mathfrak{M}_{n-\mathfrak{c}} \cong \mathfrak{M}_{n-\mathfrak{c}}'$ by Lemma 4.38. But as \mathfrak{M}' is a Kisin module, this contradicts to the assumption that G_U does not extend to R . Therefore, G as above does not extend to a p -divisible group over R . This completes the proof of Theorem 4.3.

4.4 $R = \mathcal{O}_K\langle t^{\pm 1} \rangle$ Case

We now consider the case when $R = \mathcal{O}_K\langle t^{\pm 1} \rangle$. We fix the Frobenius φ on R_0 to be given by $t \mapsto t^p$. We study various base change maps and apply the results for the cases when R is a discrete valuation ring and when $R = \mathcal{O}_K[[t]]$. An additional ingredient is the finiteness result on the moduli of connections from [36].

4.4.1 Base Changes

We consider the natural φ -equivariant map $b_g : R_0 \rightarrow R'_0 \cong W(k(t)^{\text{perf}})$ where R'_0 is the p -adic completion of $\varinjlim_{\varphi} (R_0)_{(p)}$. This induces the base change $b_g : R \rightarrow R' \cong W(k(t)^{\text{perf}}) \otimes_{W(k)} \mathcal{O}_K$. Denote $\mathfrak{S}' := R'_0[[u]]$. For $\mathfrak{q} \in \text{mSpec} R$, let $b_{\mathfrak{q}} : R_0 \rightarrow \hat{R}_{0, R_0 \cap \mathfrak{q}}$ be the natural φ -equivariant map where $\hat{R}_{0, R_0 \cap \mathfrak{q}}$ is the $(R_0 \cap \mathfrak{q})$ -adic completion of $R_{0, R_0 \cap \mathfrak{q}}$. This induces the base change $b_{\mathfrak{q}} : R \rightarrow \hat{R}_{\mathfrak{q}} \cong \hat{R}_{0, R_0 \cap \mathfrak{q}} \otimes_{W(k)} \mathcal{O}_K$, where $\hat{R}_{\mathfrak{q}}$ is the \mathfrak{q} -adic completion of $R_{\mathfrak{q}}$. By the structure theorem for complete regular local rings, we have $\hat{R}_{0, R_0 \cap \mathfrak{q}} \cong W(k_{\mathfrak{q}})[[s]]$ where $k_{\mathfrak{q}} := R_{\mathfrak{q}}/\mathfrak{q}$. Note that the Frobenius φ on $\hat{R}_{0, R_0 \cap \mathfrak{q}} \cong W(k_{\mathfrak{q}})[[s]]$ induced by $b_{\mathfrak{q}}$ does not necessarily map $s \mapsto s^p$, but it is immediate to check that all the constructions and results in Section 4.3.1 for Kisin modules and torsion Kisin modules remain to hold by the same proofs. Denote $\mathfrak{S}_{\mathfrak{q}} := \hat{R}_{0, R_0 \cap \mathfrak{q}}[[u]]$.

For \mathfrak{M} a finite \mathfrak{S}_n -module which is u -torsion free, we let $\mathfrak{M}_g := \mathfrak{M} \otimes_{\mathfrak{S}, b_g} \mathfrak{S}'$ the \mathfrak{S}'_n -

module induced by b_g , and let $\mathfrak{M}_{\mathfrak{q}} := \mathfrak{M} \otimes_{\mathfrak{S}, b_{\mathfrak{q}}} \mathfrak{S}_{\mathfrak{q}}$ induced by $b_{\mathfrak{q}}$ similarly.

Lemma 4.47. *Let \mathfrak{M} be a finite u -torsion free \mathfrak{S}_n -module equipped with φ -linear endomorphism $\varphi_{\mathfrak{M}}$ such that $\text{coker}(1 \otimes \varphi_{\mathfrak{M}})$ is killed by $E(u)^r$. Suppose that $\mathcal{M} := R_0 \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ is equipped with a topologically quasi-nilpotent integrable connection $\nabla : \mathcal{M} \rightarrow \mathcal{M} \otimes_{R_0} \hat{\Omega}_{R_0}$ commuting with Frobenius. Assume that $\mathfrak{M}[\frac{1}{u}] := \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$ with Frobenius induced from $\varphi_{\mathfrak{M}}$ is an étale $(\varphi, \mathcal{O}_{\mathcal{E}})$ -module which is projective over $\mathcal{O}_{\mathcal{E}, n}$. We define the saturation of \mathfrak{M} to be*

$$\text{Sat}(\mathfrak{M}) := \mathfrak{M}_g \bigcap \mathfrak{M}[\frac{1}{u}].$$

Then, we have $\text{Sat}(\mathfrak{M}) \in \text{Mod}_{\mathfrak{S}}^1(\varphi, \nabla)$, and $\text{Sat}(\mathfrak{M})[\frac{1}{u}] = \mathfrak{M}[\frac{1}{u}]$.

Proof. First, note that $\text{Sat}(\mathfrak{M})$ is finite over \mathfrak{S} , since $\mathfrak{M}[\frac{1}{u}]$ is finite projective over $\mathcal{O}_{\mathcal{E}, n}$ so $\text{Sat}(\mathfrak{M}) \subset u^{-l}\mathfrak{M}$ as \mathfrak{S} -modules for some integer l .

$\varphi_{\mathfrak{M}}$ on \mathfrak{M} extends naturally to a φ -semilinear endomorphism $\varphi_{\mathfrak{M}} : \text{Sat}(\mathfrak{M}) \rightarrow \text{Sat}(\mathfrak{M})$. By [6, Lemma 7.1.8], $\varphi : \mathfrak{S} \rightarrow \mathfrak{S}$ is flat. Thus, since \mathfrak{M} is u -torsion free and $\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$ is an étale $(\varphi, \mathcal{O}_{\mathcal{E}})$ -module, $1 \otimes \varphi_{\mathfrak{M}} : \varphi^*\mathfrak{M} \rightarrow \mathfrak{M}$ is injective. There exists a \mathfrak{S} -linear map $\psi_{\mathfrak{M}} : \mathfrak{M} \rightarrow \varphi^*\mathfrak{M}$ such that $(1 \otimes \varphi_{\mathfrak{M}}) \circ \psi_{\mathfrak{M}} = c_0^{-r} E(u)^r \text{Id}_{\mathfrak{M}}$. This extends naturally to $\psi_{\mathfrak{M}} : \text{Sat}(\mathfrak{M}) \rightarrow \varphi^*(\text{Sat}(\mathfrak{M}))$, and we have $(1 \otimes \varphi_{\mathfrak{M}}) \circ \psi_{\mathfrak{M}} = c_0^{-r} E(u)^r \text{Id}_{\text{Sat}(\mathfrak{M})}$. Hence, $\text{coker}(1 \otimes \varphi_{\mathfrak{M}} : \varphi^*(\text{Sat}(\mathfrak{M})) \rightarrow \text{Sat}(\mathfrak{M}))$ is killed by $E(u)^r$.

Since $\hat{\Omega}_{R_0} \cong R_0 dt$, we have the natural induced connection

$$\nabla : R_0 \otimes_{\varphi, \mathfrak{S}} \text{Sat}(\mathfrak{M}) \rightarrow (R_0 \otimes_{\varphi, \mathfrak{S}} \text{Sat}(\mathfrak{M})) \otimes_{R_0} \hat{\Omega}_{R_0}$$

which is topologically quasi-nilpotent and integrable, and commutes with Frobenius.

It remains to show that the \mathfrak{S} -projective dimension of $\text{Sat}(\mathfrak{M}) \leq 1$. For each $\mathfrak{q} \in \text{mSpec} R$, $\hat{R}_{\mathfrak{q}} \cong (W(k_{\mathfrak{q}}) \otimes_{W(k)} \mathcal{O}_K)[[s]]$, and we define the saturation of $\mathfrak{M}_{\mathfrak{q}}$ as a module over

$\mathfrak{S}_{\mathfrak{q}}$ to be $\text{Sat}(\mathfrak{M}_{\mathfrak{q}}) := \mathfrak{M}_{\mathfrak{q}}[\frac{1}{u}] \cap \mathfrak{M}_{\mathfrak{q}}[\frac{1}{s}]$ as in Lemma 4.32.

Let \tilde{k} be a composite field of $k(t)^{\text{perf}}$ and $k_{\mathfrak{q}}((s))^{\text{perf}}$. Note that

$$\{\text{Spec}(W(\tilde{k})[[u]]/p^n), \text{Spec}(\mathfrak{S}_{\mathfrak{q},n}[\frac{1}{u}][\frac{1}{s}])\}$$

is an fpqc covering of $\text{Spec}(\mathfrak{S}_{\mathfrak{q},n}[\frac{1}{s}])$. So by fpqc descent, $\text{Sat}(\mathfrak{M}) \otimes_{\mathfrak{S}} \mathfrak{S}_{\mathfrak{q}}[\frac{1}{s}] = \text{Sat}(\mathfrak{M}_{\mathfrak{q}})[\frac{1}{s}]$ since they agree over the fpqc covering. This implies by the definition of $\text{Sat}(\mathfrak{M})$ that $\text{Sat}(\mathfrak{M}) \otimes_{\mathfrak{S}, b_{\mathfrak{q}}} \mathfrak{S}_{\mathfrak{q}} = \text{Sat}(\mathfrak{M}_{\mathfrak{q}})$. Thus, the $\mathfrak{S}_{\mathfrak{q}}$ -projective dimension of $\text{Sat}(\mathfrak{M}) \otimes_{\mathfrak{S}, b_{\mathfrak{q}}} \mathfrak{S}_{\mathfrak{q}} \leq 1$ for each $\mathfrak{q} \in \text{mSpec} R$, and hence \mathfrak{S} -projective dimension of $\text{Sat}(\mathfrak{M}) \leq 1$. \square

4.4.2 Moduli of Connections

To construct a system of torsion Kisin modules with compatible connections, we use a finiteness result on the number of such connections at each level. This follows from the construction of *moduli of connections* in [36] which we recall here. In this subsection, we work over a general base ring $R = R_0 \otimes_{W(k)} \mathcal{O}_K$ where R_0 is unramified-good.

Suppose that for each $n \geq 1$, we are given $\mathfrak{N}_n \in \text{Mod}_{\mathfrak{S}}^{\text{tor},1}(\varphi, \nabla)$ a torsion Kisin module of height 1 such that \mathfrak{N}_n is projective of rank h over \mathfrak{S}/p^n and there exists a \mathfrak{S} -linear isomorphism $p\mathfrak{N}_{n+1} \cong \mathfrak{N}_n$ which is φ -equivariant but not necessarily compatible with the connections. We let $\mathfrak{N} = \varprojlim_n \mathfrak{N}_n \in \text{Mod}_{\mathfrak{S}}^1(\varphi)$ a Kisin module of height 1, and let $\mathcal{N} = R_0 \otimes_{\varphi, \mathfrak{S}} \mathfrak{N}$. Note that \mathcal{N} is a frame as defined in [23, 2.1]. Fix an R_0 -direct factor $\mathcal{N}^1 \subset \mathcal{N}$ lifting $\text{Fil}^1 \mathcal{N}/p\mathcal{N} \subset \mathcal{N}/p\mathcal{N}$, and let $\tilde{\mathcal{N}} := \varphi^*(\mathcal{N} + \frac{1}{p}\mathcal{N}^1) \subset \mathcal{N}[\frac{1}{p}]$.

By passing to a Zariski covering of $\text{Spf}(R_0, p)$, we may assume that $\mathcal{N}^1, \mathcal{N}/\mathcal{N}^1$ and $\hat{\Omega}_{R_0}$ are all free over R_0 . Fix an R_0 -basis of \mathcal{N} adapted to the direct factor \mathcal{N}^1 . For each $n \geq 1$,

we mean by a *connection* on $R_0/p^n \otimes_{R_0} \mathcal{N}$ an additive morphism

$$\nabla_n : R_0/p^n \otimes_{R_0} \mathcal{N} \rightarrow R_0/p^n \otimes_{R_0} \mathcal{N} \otimes_{R_0} \hat{\Omega}_{R_0}$$

which satisfies the Leibnitz rule (i.e., $\nabla_n(ax) = a\nabla_n(x) + x \otimes da$ for $a \in R_0/p^n$ and $x \in \mathcal{N}$).

Note that if the Frobenius φ on $R_0/p^{n+1} \otimes_{R_0} \mathcal{N}$ is horizontal for a connection ∇_{n+1} , then the induced connection ∇_n on $R_0/p^n \otimes_{R_0} \mathcal{N}$ satisfies the following commutative diagram:

$$\begin{array}{ccc} R_0/p^n \otimes_{R_0} \tilde{\mathcal{N}} & \xrightarrow{\varphi^*(\nabla_n)} & R_0/p^n \otimes_{R_0} \tilde{\mathcal{N}} \otimes_{R_0} \hat{\Omega}_{R_0} \\ \downarrow 1 \otimes \varphi_{\mathcal{N}} & & \downarrow (1 \otimes \varphi_{\mathcal{N}}) \otimes \text{id}_{\hat{\Omega}_{R_0}} \\ R_0/p^n \otimes_{R_0} \mathcal{N} & \xrightarrow{\nabla_n} & R_0/p^n \otimes_{R_0} \mathcal{N} \otimes_{R_0} \hat{\Omega}_{R_0} \end{array} \quad (4.5)$$

Here, $\varphi^*(\nabla_n)$ is defined by choosing an arbitrary lift of ∇_n over R_0/p^{n+1} , which does not depend on the choice of such a lift (cf. [36, 3.1.1(9)]).

Now, if we are given a connection ∇_n satisfying the commutative diagram (4.5), then the lifts ∇_{n+1} which satisfies (4.5) for $n+1$ correspond to the solutions of an Artin-Schreier system of equation over R_0/p in the form $\underline{x} = B\varphi(\underline{x}) + C_n$, as given in [36, 3.2 (15)]. In particular, we see from the proof of [36, Theorem 2.4.1] that the number of such lifts is finite. Hence, we have the following proposition:

Proposition 4.48. *In the situation as above, we can equip $R_0 \otimes_{\varphi, \mathfrak{S}} \mathfrak{N}$ with a topologically quasi-nilpotent integrable connection ∇ commuting with Frobenius, so that $\mathfrak{N} \in \text{Mod}_{\mathfrak{S}}^1(\varphi, \nabla)$.*

Proof. Let $\mathcal{N}_n = R_0 \otimes_{\varphi, \mathfrak{S}} \mathfrak{N}_n$. By assumption, we note that $R_0 \otimes_{\varphi, \mathfrak{S}} \mathfrak{N}_n$ is equipped with a connection $\nabla_{\mathfrak{N}_n}$ which is integrable, topologically quasi-nilpotent, and compatible with Frobenius. Denote by S_n the multiset $\{R_0/p^n \otimes_{R_0} \nabla_{\mathfrak{N}_k} \mid k \geq n+1\}$ of connections on \mathcal{N}_n . Note that for each $k \geq n+1$, $R_0/p^n \otimes_{R_0} \nabla_{\mathfrak{N}_k}$ satisfies the commutative diagram (4.5) for

n . We construct a system of connections ∇_n on \mathcal{N}_n inductively as follows. For $n = 1$, since the number of isomorphism classes of elements in S_1 is finite, we have by the Pigeon Hole Principle that there exists a connection ∇_1 on \mathcal{N}_1 which is isomorphic to infinitely many elements in the multiset S_1 . And when we are given a choice of connection ∇_n , we have that the number of isomorphism classes of elements in S_{n+1} which lift ∇_n is finite. So we can choose a connection ∇_{n+1} on \mathcal{N}_{n+1} lifting ∇_n which is isomorphic to infinitely many elements in S_{n+1} .

Now, let $\nabla = \varprojlim_n \nabla_n$ be the connection on $R_0 \otimes_{\varphi, \mathfrak{S}} \mathfrak{N}$. Then, ∇ commutes with Frobenius, and is integrable and topologically quasi-nilpotent by [36, Theorem 3.2]. \square

4.4.3 Proof of Theorem 4.5 for $R = \mathcal{O}_K\langle t^{\pm 1} \rangle$

We now complete the proof of Theorem 4.5. Let $G = (G_n)_{n \geq 0}$ be a p -divisible group over $R[\frac{1}{p}]$ of height h , and suppose that for each n , G_n extends to a finite locally free group scheme $G_{n,R}$ over R . Let $M \in \text{Mod}_{\mathcal{O}_{\mathcal{E}}}^{\text{ét}, \text{pr}}(\varphi)$ be the étale φ -module of rank h corresponding to G , and let $M_n := M/p^n \in \text{Mod}_{\mathcal{O}_{\mathcal{E}}}^{\text{ét}, \text{tor}}(\varphi)$. For each n , let $\mathfrak{M}_n := \mathfrak{M}^*(G_{n,R}) \in \text{Mod}_{\mathfrak{S}}^{\text{tor}, 1}(\varphi, \nabla)$ be the corresponding torsion Kisin module of height 1. We have $\mathfrak{M}_n \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} \cong M_n$. Let $\mathfrak{M}'_n := \text{Sat}(\ker(p^{2\mathfrak{c}} \mathfrak{M}_{n+3\mathfrak{c}} \xrightarrow{p^n} p^{n+2\mathfrak{c}} \mathfrak{M}_{n+3\mathfrak{c}}))$, where the constant \mathfrak{c} is as defined in Section 4.3.1. By Lemma 4.47, $\mathfrak{M}'_n \in \text{Mod}_{\mathfrak{S}}^1(\varphi, \nabla)$.

Suppose that $e \leq p - 1$.

Proposition 4.49. *For each $n \geq 1$, \mathfrak{M}'_n is projective over \mathfrak{S}/p^n . Furthermore, if we identify $p\mathfrak{M}'_{n+1} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} = \mathfrak{M}'_n \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$, then $p\mathfrak{M}'_{n+1} = \mathfrak{M}'_n$.*

Proof. We apply the results in Section 4.3.2 for each base change $b_{\mathfrak{q}}$ where $\mathfrak{q} \in \text{mSpec} R$. From the proof of Lemma 4.47, we see that

$$\mathfrak{M}'_{n,\mathfrak{q}} = \text{Sat}(\ker(p^{2\mathfrak{c}} \mathfrak{M}_{n+3\mathfrak{c},\mathfrak{q}} \xrightarrow{p^n} p^{n+2\mathfrak{c}} \mathfrak{M}_{n+3\mathfrak{c},\mathfrak{q}})).$$

Since $e \leq p - 1$, we then have by Lemma 4.43 and Theorem 4.15 that $\mathfrak{M}'_{n,\mathfrak{q}}$ is projective over $\mathfrak{S}_{\mathfrak{q}}/p^n$, by the same argument as in Section 4.3.2 using Lemma 4.41. Thus, \mathfrak{M}'_n is projective over \mathfrak{S}/p^n .

Considering both \mathfrak{M}'_n and $p\mathfrak{M}'_{n+1}$ as \mathfrak{S} -submodules of $\mathfrak{M}'_n \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$, let

$$\tilde{\mathfrak{M}} = \text{coker}(\mathfrak{M}'_n \hookrightarrow \mathfrak{M}'_n + p\mathfrak{M}'_{n+1}).$$

Then $\tilde{\mathfrak{M}} \otimes_{\mathfrak{S}} \mathfrak{S}_{\mathfrak{q}} = 0$ for each $\mathfrak{q} \in \text{mSpec} R$, since

$$\mathfrak{M}'_{n,\mathfrak{q}} = p\mathfrak{M}'_{n+1,\mathfrak{q}}$$

by the results in Section 4.3. Thus, $\tilde{\mathfrak{M}} = 0$ and $p\mathfrak{M}'_{n+1} \subset \mathfrak{M}'_n$. We similarly have $\mathfrak{M}'_n \subset p\mathfrak{M}'_{n+1}$, and hence $p\mathfrak{M}'_{n+1} = \mathfrak{M}'_n$. \square

Now, by applying Lemma 4.39 for each base change $b_{\mathfrak{q}}$ with $\mathfrak{q} \in \text{mSpec} R$, we see from the construction of \mathfrak{M}'_n that the finite locally free group scheme corresponding to \mathfrak{M}'_n agrees with G_n over $R[\frac{1}{p}]$. Thus, by Proposition 4.48, G extends to a p -divisible group over R . The uniqueness follows from [35, Theorem 4]. \square

4.5 Higher Dimensional Case with $e < p - 1$ and Barsotti-Tate Representations

4.5.1 Higher Dimensional Case with $e < p - 1$

We now study the case when the base ring has a higher dimension and the ramification index $e < p - 1$. We first recall the following Weierstrass Preparation Theorem proved in [38]. For $\mathfrak{S} = W(k)[[t_1, \dots, t_d]][[u]]$ and $f = \sum_{i \geq 0} a_i(\underline{t})u^i \in \mathfrak{S}_n$ with $a_i(\underline{t}) \in W(k)[[t_1, \dots, t_d]]/p^n$,

we define the u -order of f to be

$$\text{ord}_u(f) := \min\{i \mid a_i(\underline{t}) \in W(k)[[t_1, \dots, t_d]]/p^n \text{ is a unit}\}.$$

Theorem 4.50. (cf. [38, Corollary 3.2]) *Let $\mathfrak{S} = W(k)[[t_1, \dots, t_d]][u]$, and let $f \in \mathfrak{S}_n$ with $\text{ord}_u(f) = l$. Then, there exists a unit $\epsilon \in \mathfrak{S}_n$ and a polynomial $F \in (W(k)[[t_1, \dots, t_d]]/p^n)[u]$ of $\deg_u(F) = l$ such that $F \equiv u^l \pmod{(p, t_1, \dots, t_d)}$ and $f = \epsilon F$.*

Proof. This is a special case of [38, Corollary 3.2]. \square

Theorem 4.51. *Let $R = \mathcal{O}_K[[t_1, \dots, t_d]]$ or $R = \mathcal{O}_K\langle t_1^{\pm 1}, \dots, t_d^{\pm 1} \rangle$, and suppose that $e < p - 1$. Let $G = (G_n)_{n \geq 1}$ be a p -divisible group over $R[\frac{1}{p}]$ such that for each n , G_n extends to a finite locally free group scheme $G_{n,R}$ over R . Then G extends to a p -divisible group over R uniquely up to isomorphism.*

Proof. Let $b_g : R_0 \rightarrow R'_0 \cong W((\text{Frac}(R_0/p))^{\text{perf}})$ be the natural φ -equivariant map, where R'_0 is the p -adic completion of $\varinjlim_{\varphi} (R_0)_{(p)}$. We then have the induced base change $b_g : R \rightarrow W((\text{Frac}(R_0/p))^{\text{perf}}) \otimes_{W(k)} \mathcal{O}_K$. Let $M \in \text{Mod}_{\mathcal{O}_{\mathcal{E}}}^{\text{ét, pr}}(\varphi)$ be the étale $(\varphi, \mathcal{O}_{\mathcal{E}})$ -module corresponding to G , and let $M_n := M/p^n \in \text{Mod}_{\mathcal{O}_{\mathcal{E}}}^{\text{ét, tor}}(\varphi)$. For each n , let $\mathfrak{M}_n := \mathfrak{M}^*(G_{n,R}) \in \text{Mod}_{\mathfrak{S}}^{\text{tor, 1}}(\varphi, \nabla)$ be the torsion Kisin module of height 1. We have $\mathfrak{M}_n \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} \cong M_n$.

We first consider the case $R = \mathcal{O}_K[[t_1, \dots, t_d]]$, which we equip with the Frobenius given by $t_i \mapsto t_i^p$. Let $b_0 : R_0 \rightarrow W(k)$ be the base change given by $t_i \mapsto 0$. Let I_k be the k -th Fitting ideal of \mathfrak{M}_n over \mathfrak{S}_n . Let $I_{k,g}$ and $I_{k,0}$ be the k -th Fitting ideal of $\mathfrak{M}_n \otimes_{\mathfrak{S}, b_g} W((\text{Frac}(R_0/p))^{\text{perf}})[[u]]$ over $W((\text{Frac}(R_0/p))^{\text{perf}})[[u]]/p^n$ and the k -th Fitting ideal of $\mathfrak{M}_n \otimes_{\mathfrak{S}, b_0} W(k)[[u]]$ over $W(k)[[u]]/p^n$ respectively. Note that $I_{k,g}$ and $I_{k,0}$ are given by the images of I_k under the corresponding base change maps.

Let h be the height of G . From the uniqueness of finite flat group scheme models over a discrete valuation ring of mixed characteristics with $e < p - 1$, we see that

$\mathfrak{M}_n \otimes_{\mathfrak{S}, b_g} W((\text{Frac}(R_0/p))^{\text{perf}})[[u]]$ is free of rank h over $W((\text{Frac}(R_0/p))^{\text{perf}})[[u]]/p^n$, and $\mathfrak{M}_n \otimes_{\mathfrak{S}, b_0} W(k)[[u]]$ is free of rank h over $W(k)[[u]]/p^n$. Thus,

$$I_{k,g} = \begin{cases} 0 & \text{if } k < h \\ W((\text{Frac}(R_0/p))^{\text{perf}})[[u]]/p^n & \text{if } k \geq h, \end{cases}$$

and

$$I_{k,0} = \begin{cases} 0 & \text{if } k < h \\ W(k)[[u]]/p^n & \text{if } k \geq h. \end{cases}$$

This implies

$$I_k = \begin{cases} 0 & \text{if } k < h \\ \mathfrak{S}_n & \text{if } k \geq h, \end{cases}$$

which shows that \mathfrak{M}_n is projective of rank h over \mathfrak{S}_n .

Now, identify $p\mathfrak{M}_{n+1} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} = M_n = \mathfrak{M}_n \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$, and consider both \mathfrak{M}_n and $p\mathfrak{M}_{n+1}$ as \mathfrak{S}_n -submodules of M_n . \mathfrak{M}_n and $p\mathfrak{M}_{n+1}$ are free over \mathfrak{S}_n , and we have

$$\mathfrak{M}_n \otimes_{\mathfrak{S}, b_g} W((\text{Frac}(R_0/p))^{\text{perf}})[[u]] = p\mathfrak{M}_{n+1} \otimes_{\mathfrak{S}, b_g} W((\text{Frac}(R_0/p))^{\text{perf}})[[u]] \quad (4.6)$$

and

$$\mathfrak{M}_n \otimes_{\mathfrak{S}, b_0} W(k)[[u]] = p\mathfrak{M}_{n+1} \otimes_{\mathfrak{S}, b_0} W(k)[[u]] \quad (4.7)$$

from the uniqueness result for the discrete valuation ring cases. Let (e_1, \dots, e_h) (resp. (e'_1, \dots, e'_h)) be a \mathfrak{S}_n -basis for \mathfrak{M}_n (resp. $p\mathfrak{M}_{n+1}$) inside M_n . Since $p\mathfrak{M}_{n+1} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} = M_n = \mathfrak{M}_n \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$, we have

$$(e_1, \dots, e_h) = (e'_1, \dots, e'_h)A$$

where A is a $h \times h$ matrix with coefficients in $\mathcal{O}_{\mathcal{E}, n}$ such that $\det A$ is a unit in $\mathcal{O}_{\mathcal{E}, n}$. By (4.6),

A has coefficients in $\mathcal{O}_{\mathcal{E},n} \cap (W((\text{Frac}(R_0/p))^{\text{perf}})[[u]]/p^n) = \mathfrak{S}_n$. We can assume $\det A = u^l$ by Theorem 4.50. We then have $l = 0$ by (4.7), and $p\mathfrak{M}_{n+1} = \mathfrak{M}_n$. Thus, $\mathfrak{M} := \varprojlim_n \mathfrak{M}_n$ is a Kisin module of height 1, whose corresponding p -divisible group over R extends G .

Now we consider the case $R = \mathcal{O}_K \langle t_1^{\pm 1}, \dots, t_d^{\pm 1} \rangle$. For each $\mathfrak{q} \in \text{mSpec} R$, let $b_{\mathfrak{q}} : R_0 \rightarrow \hat{R}_{0,R_0 \cap \mathfrak{q}}$ be the natural φ -equivariant map where $\hat{R}_{0,R_0 \cap \mathfrak{q}}$ is the $(R_0 \cap \mathfrak{q})$ -adic completion of $R_{0,R_0 \cap \mathfrak{q}}$. This induces the base change $b_{\mathfrak{q}} : R \rightarrow \hat{R}_{\mathfrak{q}} \cong \hat{R}_{0,R_0 \cap \mathfrak{q}} \otimes_{W(k)} \mathcal{O}_K$, where $\hat{R}_{\mathfrak{q}}$ is the \mathfrak{q} -adic completion of $R_{\mathfrak{q}}$. By the structure theorem for complete regular local rings, we have $\hat{R}_{0,R_0 \cap \mathfrak{q}} \cong W(k_{\mathfrak{q}})[[s_1, \dots, s_d]]$ where $k_{\mathfrak{q}} := R_{\mathfrak{q}}/\mathfrak{q}$. Denote $\mathfrak{S}_{\mathfrak{q}} := \hat{R}_{0,\mathfrak{q} \cap R_0}[[u]]$.

By the above result for the case of a formal power series over a discrete valuation ring, $\mathfrak{M} \otimes_{\mathfrak{S}, b_{\mathfrak{q}}} \mathfrak{S}_{\mathfrak{q}}$ is projective over $\mathfrak{S}_{\mathfrak{q}}/p^n$ for each $\mathfrak{q} \in \text{mSpec} R$. Thus, \mathfrak{M}_n is projective over \mathfrak{S}_n . Identify $p\mathfrak{M}_{n+1} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} = M_n = \mathfrak{M}_n \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$, and consider both \mathfrak{M}_n and $p\mathfrak{M}_{n+1}$ as \mathfrak{S}_n -submodules of M_n (forgetting the connections). Let $\tilde{\mathfrak{M}} = \text{coker}(\mathfrak{M}_n \hookrightarrow \mathfrak{M}_n + p\mathfrak{M}_{n+1})$. Then $\tilde{\mathfrak{M}} \otimes_{\mathfrak{S}} \mathfrak{S}_{\mathfrak{q}} = 0$ for each $\mathfrak{q} \in \text{mSpec} R$, by the result above for the formal power series base ring. Thus, $p\mathfrak{M}_{n+1} = \mathfrak{M}_n$ as φ -modules. By Proposition 4.48, G extends to a p -divisible group over R .

In both cases, the uniqueness follows from [35, Theorem 4]. \square

Above proof can be easily generalized to show the following theorem.

Theorem 4.52. *Let \mathcal{X} be a (formally) smooth geometrically connected scheme over \mathcal{O}_K with $e < p - 1$. Suppose that \mathcal{X} can be covered by finitely many affine open subschemes $\text{Spec} A_i$ such that A_i is good.*

Let $G = (G_n)_{n \geq 1}$ be a p -divisible group over $\mathcal{X} \times_{\mathbb{Z}_p} \mathbb{Q}_p$ such that each G_n extends to a finite locally free group scheme $G_{n,\mathcal{X}}$ over \mathcal{X} . Then G extends to a p -divisible group over \mathcal{X} .

Proof. Note that for each maximal ideal $\mathfrak{q} \subset A_i$, by Lemma 4.7 and the structure theorem of

complete regular local rings, the \mathfrak{q} -adic completion of $A_{i,\mathfrak{q}}$ is a formal power series ring over $W(k_{\mathfrak{q}}) \otimes_{W(k)} \mathcal{O}_K$ where $k_{\mathfrak{q}} := A_i/\mathfrak{q}$. By Theorem 4.51 and faithfully flat descent, $G_{n,\mathcal{X}}[p^i]$ is a locally free group scheme over \mathcal{X} for each n and i . From the proof of Theorem 4.51, we see that $\mathfrak{M}^*(G_{n,\mathcal{X}}[p^i] \times_{\mathcal{X}} \text{Spec} A_j) = \mathfrak{M}^*(G_{n+1,\mathcal{X}}[p^i] \times_{\mathcal{X}} \text{Spec} A_j)$ as φ -modules (forgetting the connections) for each n, i, j . Thus, the result follows from Proposition 4.48. \square

4.5.2 Relation to Barsotti-Tate Representations in the Relative Case

We study how the above results are related to Barsotti-Tate representations. Let \mathcal{X} be a (formally) smooth geometrically connected scheme over \mathcal{O}_K , and let \mathcal{G} be the étale fundamental group of $\mathcal{X} \times_{\mathcal{O}_K} K$. Recall that we say a \mathbb{Q}_p -representation of \mathcal{G} is *Barsotti-Tate* if it arises from a p -divisible group over \mathcal{X} . For a \mathbb{Z}_p -representation L of \mathcal{G} , we say it is *Barsotti-Tate* if $L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is Barsotti-Tate.

Proposition 4.53. *Suppose $e \leq p - 1$ and \mathcal{X} has dimension 2. Suppose further that \mathcal{X} satisfies the assumption as in Theorem 4.52.*

If a \mathbb{Z}_p -representation L of \mathcal{G} is Barsotti-Tate, then there exists a p -divisible group G over \mathcal{X} such that $T_p(G) \cong L$ as $\mathbb{Z}_p[\mathcal{G}]$ -modules.

Proof. Since L is Barsotti-Tate, there exists a p -divisible group G' over \mathcal{X} such that $L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong T_p(G') \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ as \mathbb{Q}_p -representations of \mathcal{G} . Let $\tilde{G}' := G' \times_{\mathcal{O}_K} K$, and let L' be the \mathbb{Z}_p -representation of \mathcal{G} corresponding to \tilde{G}' . Let \tilde{G} be the p -divisible group over $\mathcal{X} \times_{\mathcal{O}_K} K$ corresponding to L .

Since $p^n L \subset L'$ and $p^n L' \subset L$ for some positive integer n , we have an isogeny $f : \tilde{G}' \rightarrow \tilde{G}$. Let $\tilde{H} := \ker(f)$, which is a finite locally free group scheme over $\mathcal{X} \times_{\mathcal{O}_K} K$. Then, we have a closed embedding $h : \tilde{H} \hookrightarrow \tilde{G}'[p^m]$ for some positive integer m . Note that $\tilde{G}'[p^m]$ extends to a finite locally free group scheme $G'[p^m]$ over \mathcal{X} .

Let H be the scheme theoretic closure over \mathcal{X} obtained from h , as in [32, Section 2]. We claim that H is a finite locally free group scheme over \mathcal{X} . For this, consider an affine open subscheme $\text{Spec} A_i \subset \mathcal{X}$ as in the assumption in Theorem 4.52. For each maximal ideal $\mathfrak{q} \subset A_i$, denote by $\hat{A}_{i,\mathfrak{q}}$ the \mathfrak{q} -adic completion of $A_{i,\mathfrak{q}}$. By the result for the case of discrete valuation base rings and Lemma 4.38 and 4.43 using $e \leq p-1$, we see that $H \times_{\mathcal{X}} \text{Spec} \hat{A}_{i,\mathfrak{q}}$ is a finite locally free group scheme. Thus, by faithfully flat descent, H is a finite locally free group scheme.

h induces a closed embedding $H \hookrightarrow G'[p^m]$, and $G := G'/H$ is a p -divisible group over \mathcal{X} . It is clear from the construction that $T_p(G) \cong L$ as $\mathbb{Z}_p[\mathcal{G}]$ -modules. \square

As an immediate corollary to Proposition 4.53, we have the following.

Theorem 4.54. *Let $e < p-1$, and suppose that \mathcal{X} has dimension 2 and satisfies the assumption in Theorem 4.52. Let L be a \mathbb{Z}_p -representation of \mathcal{G} . Suppose that for each positive integer n , L/p^n is torsion Barsotti-Tate, in the sense that there exist \mathcal{G} -stable \mathbb{Z}_p -lattices $T'_n \subset T_n$ inside a Barsotti-Tate \mathbb{Q}_p -representation such that $L/p^n \cong T_n/T'_n$ as $\mathbb{Z}_p[\mathcal{G}]$ -modules.*

Then, L is Barsotti-Tate.

Proof. Let \tilde{G} be the p -divisible group over $\mathcal{X} \times_{\mathcal{O}_K} K$ corresponding to L . Note that $\tilde{G}[p^n]$ is the finite locally free group scheme over $\mathcal{X} \times_{\mathcal{O}_K} K$ corresponding to L/p^n . Since T_n, T'_n are Barsotti-Tate, we see from Proposition 4.53 and its proof that there exist p -divisible groups G_n^1, G_n^2 over \mathcal{X} with an isogeny $f_n : G_n^2 \rightarrow G_n^1$ such that $T_p(G_n^1) \cong T_n$ and $T_p(G_n^2) \cong T'_n$ as $\mathbb{Z}_p[\mathcal{G}]$ -modules and that f_n induces the inclusion $T'_n \subset T_n$.

$\ker(f_n)$ is a finite locally free group scheme over \mathcal{X} which extends $\tilde{G}[p^n]$. Thus, by Theorem 4.52, \tilde{G} extends to a p -divisible group G over \mathcal{X} . Since $T_p(G) \cong L$ as $\mathbb{Z}_p[\mathcal{G}]$ -modules, L is Barsotti-Tate. \square

Corollary 4.55. *Let $e < p - 1$, and suppose that \mathcal{X} has dimension 2 and satisfies the assumption in Theorem 4.52. Then being Barsotti-Tate as a representation of \mathcal{G} is a Zariski-closed condition.*

Proof. It follows formally by applying the Schlessinger's criteria (cf. Theorem 3.29 and [31, Proposition 1.2]). \square

Theorem 4.56. *Let $e \geq p$ and $\mathcal{X} = \operatorname{Spec} \mathcal{O}_K[[t]]$. Then, there exists a \mathbb{Z}_p -representation L of \mathcal{G} such that L/p^n is torsion Barsotti-Tate for each positive integer n but L is not Barsotti-Tate.*

Proof. By Theorem 4.3, there exists a p -divisible group \tilde{G} over $\mathcal{X} \times_{\mathcal{O}_K} K$ such that each $\tilde{G}[p^n]$ extends to a finite locally free group scheme over \mathcal{X} but \tilde{G} does not extend to a p -divisible group over \mathcal{X} . Let L be the \mathbb{Z}_p -representation of \mathcal{G} corresponding to \tilde{G} . Since every finite locally free group scheme over \mathcal{X} embeds into a p -divisible group, L/p^n is torsion Barsotti-Tate for each n . However, L is not Barsotti-Tate since \tilde{G} does not extend over \mathcal{X} . \square

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